

UNIVERSALITY OF TWO-DIMENSIONAL CRITICAL CELLULAR AUTOMATA

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ABSTRACT. We study the class of monotone, two-state, deterministic cellular automata, in which sites are activated (or ‘infected’) by certain configurations of nearby infected sites. These models have close connections to statistical physics, and several specific examples have been extensively studied in recent years by both mathematicians and physicists. This general setting was first studied only recently, however, by Bollobás, Smith and Uzzell, who showed that the family of all such ‘bootstrap percolation’ models on \mathbb{Z}^2 can be naturally partitioned into three classes, which they termed subcritical, critical and supercritical.

In this paper we determine the order of the threshold for percolation (complete occupation) for every critical bootstrap percolation model in two dimensions. This ‘universality’ theorem includes as special cases results of Aizenman and Lebowitz, Gravner and Griffeath, Mountford, and van Enter and Hulshof, significantly strengthens bounds of Bollobás, Smith and Uzzell, and complements recent work of Balister, Bollobás, Przykucki and Smith on subcritical models.

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Date: July 23, 2015.

2010 Mathematics Subject Classification. Primary 60K35; Secondary 60C05.

Key words and phrases. cellular automata, bootstrap percolation, universality, critical probability, metastability.

B.B. is partially supported by NSF grant DMS 1301614 and MULTIPLEX grant no. 317532, R.M. by CNPq (Proc. 479032/2012-2 and Proc. 303275/2013-8), and P.S. by a CNPq bolsa PDJ.

1. INTRODUCTION

An important and challenging problem in statistical physics, probability theory and combinatorics is to understand the typical global behaviour of so-called ‘lattice models’, including cellular automata, percolation models, and spin models. Although these models are defined in terms of local interactions between the sites of the lattice, it is typically observed in simulations that, in fixed dimensions, the macroscopic behaviour of the models does not seem to depend on the precise nature of these local interactions. Indeed, since the breakthrough work of Kadanoff [30] and the development of the renormalization group framework by Wilson [39], this phenomenon of *universality* has been widely predicted to occur throughout statistical physics (see, for example, [21, 31, 34]). Despite this, it has been proved rigorously in only a small handful of cases. One example of a model for which universality is partially understood is the Ising model, for which it was proved recently that the critical exponents exist and are equal on a large class of planar graphs [18, 22].

Cellular automata are interacting particle systems whose update rules are local and homogeneous. Since their introduction by von Neumann [38] almost 50 years ago, many particular such systems have been investigated, but no general theory has been developed for their study, and for many simple examples surprisingly little is known. In this paper we develop such a general theory for monotone, two-dimensional cellular automata with random initial configurations, which may also be thought of as monotone versions of the Glauber dynamics of the Ising model with arbitrary local interactions. The study of this extremely general class of models was initiated only recently by Bollobás, Smith and Uzzell [12], although many special cases had been studied earlier, beginning with the work of Chalupa, Leath and Reich [17] in 1979. We refer to these models as *bootstrap percolation*, but we emphasize that they are vastly more general than the family of models that usually bears this name.

The class of models we study is defined as follows. Fix $d \in \mathbb{N}$ and let $\mathcal{U} = \{X_1, \dots, X_m\}$ be an arbitrary finite collection of finite subsets of $\mathbb{Z}^d \setminus \{0\}$. We call \mathcal{U} the *update family* of the process, each $X \in \mathcal{U}$ an *update rule*, and the process itself *\mathcal{U} -bootstrap percolation*. Let the lattice Λ be either \mathbb{Z}^d or \mathbb{Z}_n^d (the d -dimensional discrete torus). Now given a set $A \subset \Lambda$ of initially *infected* sites, set $A_0 = A$, and define for each $t \geq 0$,

$$A_{t+1} = A_t \cup \{x \in \Lambda : x + X \subset A_t \text{ for some } X \in \mathcal{U}\}.$$

Thus, a site x becomes infected at time $t + 1$ if the translate by x of one of the sets of the update family is already entirely infected at time t , and infected sites remain infected forever. The set of eventually infected sites is the *closure* of A , denoted by $[A] = \bigcup_{t \geq 0} A_t$. We say that A *percolates* if $[A] = \Lambda$.

As mentioned above, this model was first introduced (in its full generality) only recently, in [12], although various special cases of it were introduced and studied

much earlier by several different authors; see for example [9, 13, 14, 17, 19, 20, 23, 24, 28]. Indeed, the general class of \mathcal{U} -bootstrap percolation models is easily seen to include as specific examples all previously studied bootstrap percolation models on lattice graphs. For example, the update family of the classical r -neighbour model on \mathbb{Z}^d , the most well-studied of all models [1, 3, 4, 15, 16, 27], consists of the $\binom{2d}{r}$ r -subsets of the $2d$ nearest neighbours of the origin. The r -neighbour models are themselves examples of *threshold models*, which, in their full generality, consist of the r -element subsets of an arbitrary finite set $Y \subset \mathbb{Z}^d \setminus \{0\}$, although, with a single exception [33], only centrally symmetric sets Y had been studied before the work of [12]. The lack of symmetry in the general setting causes all previously-developed techniques to break down, and overcoming this obstacle is one of the main tasks of this paper.

Motivated by applications to statistical physics, we shall study the global behaviour of the \mathcal{U} -bootstrap process acting on random initial sets. Specifically, let us say that a set $A \subset \Lambda$ is p -random if each of the sites of Λ is included in A independently with probability p . The key question is that of how likely it is that a random set A percolates on the lattice Λ ; in particular, one would like to know how large p must be before percolation becomes likely. The point at which this phase transition occurs is measured by the *critical probability*,

$$p_c(\Lambda, \mathcal{U}) := \inf \left\{ p : \mathbb{P}_p(A \text{ percolates in } \mathcal{U}\text{-bootstrap percolation on } \Lambda) \geq 1/2 \right\},$$

where \mathbb{P}_p denotes the product probability measure on Λ with density p .¹

For the r -neighbour model on \mathbb{Z}_n^d , with d fixed and $n \rightarrow \infty$, a great deal is known about the critical probability. Up to a constant factor, the threshold was determined by Aizenman and Lebowitz [1] for $r = 2$, by Cerf and Cirillo [15] for $d = r = 3$, and by Cerf and Manzo [16] for all remaining $2 \leq r \leq d$. The constant factor was later improved to a $1 + o(1)$ factor by Holroyd [27] in the case $d = 2$, by Balogh, Bollobás and Morris [4] for $d = 3$, and by Balogh, Bollobás, Duminil-Copin and Morris [3] for all $d \geq 4$. The r -neighbour model has also attracted attention on lattices with the dimension d tending to infinity (for example the hypercube) [5, 6], and on graphs other than lattices, including trees [7, 10] and random graphs [8, 29].

For lattice models other than the r -neighbour model, considerably less is known. Among the exceptions are two-dimensional symmetric, balanced² threshold models, for which the critical probability was determined up to a constant factor by Gravner and Griffeath [24], and asymptotically by Duminil-Copin and Holroyd [19]. Some results about the critical probabilities of a rather limited number of so-called unbalanced models are also known, which were proved by Mountford [33], van Enter and Hulshof [37], Duminil-Copin and van Enter [20], all in two dimensions, and by van Enter and Fey [36] in three dimensions.

¹Thus a p -random set is one chosen according to the distribution \mathbb{P}_p .

²These terms are defined below.

For the remainder of the paper, with the exception of a brief discussion of higher dimensions in Section 9, we restrict our attention to the case $d = 2$. As we shall see shortly, one of the key properties of the \mathcal{U} -bootstrap process is that its rough global behaviour depends only on the action of the process on discrete half-planes. In order to make this statement precise, let us introduce a little notation. For each $u \in S^1$, let $\mathbb{H}_u := \{x \in \mathbb{Z}^2 : \langle x, u \rangle < 0\}$ be the discrete half-plane whose boundary is perpendicular to u . We say that u is a *stable direction* if $[\mathbb{H}_u] = \mathbb{H}_u$ and we denote by $\mathcal{S} = \mathcal{S}(\mathcal{U}) \subset S^1$ the collection of stable directions.

The following classification of two-dimensional update families was proposed by Bollobás, Smith and Uzzell [12].

Definition 1.1. An update family \mathcal{U} is:

- *subcritical* if every semicircle in S^1 has infinite intersection with \mathcal{S} ;
- *critical* if there exists a semicircle in S^1 that has finite intersection with \mathcal{S} , and if every open semicircle in S^1 has non-empty intersection with \mathcal{S} ;
- *supercritical* if there exists an open semicircle in S^1 that is disjoint from \mathcal{S} .

The justification of the above definition was completed in two stages. First, in their original paper, Bollobás, Smith and Uzzell [12] proved that the critical probabilities of supercritical families are polynomial, while those of critical families are polylogarithmic. Later, Balister, Bollobás, Przykucki and Smith [2] proved that the critical probabilities of subcritical models are bounded away from zero. The combination of the results of [12] and [2] may be summarized as follows³:

- if \mathcal{U} is subcritical then $\liminf_{n \rightarrow \infty} p_c(\mathbb{Z}_n^2, \mathcal{U}) > 0$;
- if \mathcal{U} is critical then $p_c(\mathbb{Z}_n^2, \mathcal{U}) = (\log n)^{-\Theta(1)}$;
- if \mathcal{U} is supercritical then $p_c(\mathbb{Z}_n^2, \mathcal{U}) = n^{-\Theta(1)}$.

In this paper we significantly strengthen the bounds of [12] by determining the threshold $p_c(\mathbb{Z}_n^2, \mathcal{U})$ up to a constant factor for every critical update family. This result, which may be thought of as a universality statement for two-dimensional critical bootstrap percolation, was previously known only in the case of one very restrictive subclass of critical models [1, 24], namely the symmetric, balanced threshold models, and just two other specific models [33, 37].

The form of the threshold function depends on two properties of \mathcal{U} : the ‘difficulty’ of \mathcal{U} , and whether or not \mathcal{U} is ‘balanced’. In order to explain what these terms mean, first we need a quantitative measure of how hard it is to grow in each direction.

³Our asymptotic notation is mostly standard; however, for the avoidance of ambiguity, it is defined precisely in Section 2.4.

Let $\mathbb{Q}_1 \subset S^1$ denote the set of rational directions on the circle⁴, and for each $u \in \mathbb{Q}_1$, let ℓ_u^+ be the subset of the line $\ell_u := \{x \in \mathbb{Z}^2 : \langle x, u \rangle = 0\}$ consisting of the origin and the sites to the right of the origin as one looks in the direction of u . Similarly, let $\ell_u^- := (\ell_u \setminus \ell_u^+) \cup \{0\}$ consist of the origin and the sites to the left of the origin. Note that the line ℓ_u is infinite for every $u \in \mathbb{Q}_1$.

Definition 1.2. Given $u \in \mathbb{Q}_1$, the *difficulty* of u is

$$\alpha(u) := \begin{cases} \min \{\alpha^+(u), \alpha^-(u)\} & \text{if } \alpha^+(u) < \infty \text{ and } \alpha^-(u) < \infty \\ \infty & \text{otherwise,} \end{cases}$$

where $\alpha^+(u)$ (respectively $\alpha^-(u)$) is defined to be the minimum (possibly infinite) cardinality of a set $Z \subset \mathbb{Z}^2$ such that $[\mathbb{H}_u \cup Z]$ contains infinitely many sites of ℓ_u^+ (respectively ℓ_u^-). It follows from simple properties of stable sets (see Section 2.5) that $\alpha(u) > 0$ if and only if u is a stable direction. Now let \mathcal{C} denote the collection of open semicircles of S^1 . We define the *difficulty* of \mathcal{U} to be

$$\alpha = \alpha(\mathcal{U}) := \min_{C \in \mathcal{C}} \max_{u \in C} \alpha(u). \quad (1)$$

In Section 2 we discuss why these definitions of the difficulty of a direction under the action of \mathcal{U} and of the difficulty of \mathcal{U} itself are the natural ones. The final definition we need is as follows.

Definition 1.3. A critical update family \mathcal{U} is *balanced* if there exists a closed semicircle C such that $\alpha(u) \leq \alpha$ for all $u \in C$. It is said to be *unbalanced* otherwise.

The distinction between the open semicircles in the definition of α and the closed semicircles in the definition of balanced is subtle but important. It turns out that growth under the action of balanced critical families is completely two-dimensional, while growth under the action of unbalanced critical families is asymptotically one-dimensional. Despite this, the analysis of unbalanced families presents by far the greater number of difficulties.

We first state our main theorem in terms of the infection time of the origin. This version of the theorem is essentially equivalent to the version stated in terms of critical probabilities (which we state afterwards), and follows from the same proofs. Given $A \subset \mathbb{Z}^2$, define the stopping time

$$\tau = \tau(A, \mathcal{U}) := \min \{t \geq 0 : \mathbf{0} \in A_t\}$$

to be the time at which the origin is infected in the \mathcal{U} -bootstrap process on \mathbb{Z}^2 with $A_0 = A$. We use the expression ‘with high probability’ to mean ‘with probability tending to 1’.

⁴That is, the set of all $u \in S^1$ such that u has rational or infinite gradient with respect to the standard basis vectors.

The following theorem is the main result of this paper.

Theorem 1.4. *Let \mathcal{U} be a critical two-dimensional bootstrap percolation update family, and let A be a p -random subset of \mathbb{Z}^2 .*

(i) *If \mathcal{U} is balanced, then, with high probability as $p \rightarrow 0$,*

$$p^\alpha \log \tau = \Theta(1).$$

(ii) *If \mathcal{U} is unbalanced, then, with high probability as $p \rightarrow 0$,*

$$p^\alpha \left(\log \frac{1}{p} \right)^{-2} \log \tau = \Theta(1).$$

As mentioned above, the next theorem is essentially equivalent to Theorem 1.4.

Theorem 1.5. *Let \mathcal{U} be a critical two-dimensional bootstrap percolation update family.*

(i) *If \mathcal{U} is balanced, then*

$$p_c(\mathbb{Z}_n^2, \mathcal{U}) = \Theta\left(\frac{1}{\log n}\right)^{1/\alpha}.$$

(ii) *If \mathcal{U} is unbalanced, then*

$$p_c(\mathbb{Z}_n^2, \mathcal{U}) = \Theta\left(\frac{(\log \log n)^2}{\log n}\right)^{1/\alpha}.$$

We noted earlier that various special cases of Theorems 1.4 and 1.5 have already been proved in the literature. The critical probability for the 2-neighbour model was established by Aizenman and Lebowitz [1] using methods that have become central to the study of bootstrap percolation, including the ‘rectangles process’ and the notion of a ‘critical droplet’ (see Section 2 for details). Mountford [33] determined the critical probability of the Duarte model, which is the unbalanced threshold model consisting of all two-element subsets of

$$\{(-1, 0), (0, 1), (0, -1)\}.$$

His proof was based on martingale techniques, a fact that makes it unique among proofs of this type of theorem. Gravner and Griffeath [24] generalized the result of Aizenman and Lebowitz to the class of balanced, symmetric threshold models, using somewhat non-rigorous methods. The critical probability for one further unbalanced model, namely the one consisting of all three-element subsets of

$$\{(-2, 0), (-1, 0), (0, 1), (0, -1), (1, 0), (2, 0)\},$$

was determined by van Enter and Hulshof [37], correcting an assertion of Gravner and Griffeath [24]. Until now, the models studied by Mountford [33] and by van

Enter and Hulshof [37] were the only two unbalanced models whose critical probabilities were known, and they were, respectively, the unique such examples of ‘drift’ and ‘non-drift’ unbalanced models⁵.

One property that all of these previously studied models share, and one that simplifies the problem enormously, is that of symmetry. In all but the Duarte model, the symmetry is particularly strong, in that $X \in \mathcal{U}$ if and only if $-X \in \mathcal{U}$. The symmetry of the Duarte model is weaker (the useful property is that there exists a parallelogram of stable directions $\{u, -u, v, -v\} \subset \mathcal{S}$), but it is enough to make a significant difference to the proof. An important aspect of the general models that we study – perhaps *the* most important aspect – is the lack of any symmetry assumptions. Indeed, it is little exaggeration to say that the main task of this paper (as was that of [12]) is to handle the lack of symmetry, which causes all previously known techniques to break down.

In the cases of the 2-neighbour model of Aizenman and Lebowitz, the symmetric, balanced threshold models of Gravner and Griffeath, and the unbalanced model of van Enter and Hulshof, the critical probability has now been determined up to a $1 + o(1)$ factor. These results are due to Holroyd [27], Duminil-Copin and Holroyd [19], and Duminil-Copin and van Enter [20], respectively, and in some cases, even sharper results are known [25, 32]. Obtaining similarly sharp bounds for the general model is likely to be an important, but extremely difficult, direction for future research.

The organization of the rest of the paper is as follows. In Section 2 we give an outline of the proof, we introduce some notation, and we recall a number of basic facts about \mathcal{U} -bootstrap percolation from [12]. In Section 3 we lay the groundwork for the proofs of the upper bounds of Theorem 1.4, which are then proved in Sections 4 (balanced case) and 5 (unbalanced case). In Section 6 we define three different notions of ‘approximately internally filled’ sets and prove a number of deterministic properties of such sets. In Section 7 we deduce the lower bound in the balanced case. The hardest part of the proof is the lower bound in the unbalanced case, which is contained in Section 8. Finally, we end the paper with some open problems, including a discussion of the problem in higher dimensions.

2. OUTLINE OF THE PROOF

Let us begin by explaining why $\alpha(u)$ is the right definition of the difficulty of growing in a direction $u \in S^1$. The key fact is that there is a sense (which is formalized in Lemma 3.4) in which $\alpha(u)$ measures how hard it is to infect an entire new line in direction u , rather than merely an infinite subset of the line. More specifically, while the definition of $\alpha(u)$ only guarantees that there exist sets of $\alpha(u)$

⁵These terms are explained in Section 2, but roughly speaking, the term ‘drift’ refers to the phenomenon that occurs when $u \in \mathcal{S}$ is such that $\alpha(u) = \infty$ but $\min \{\alpha^-(u), \alpha^+(u)\} < \infty$, which in certain cases causes the growth to be biased in one direction.

sites that will infect infinitely many new sites on the line ℓ_u (with the help of \mathbb{H}_u), one can show that only boundedly many copies of this set are needed to infect the whole line. (This is false without the condition that both $\alpha^-(u)$ and $\alpha^+(u)$ are finite.)

Next let us see why the quantity $\alpha = \alpha(\mathcal{U})$ defined in (1) is the constant one should expect to see in the exponent of the critical probability in Theorem 1.5. In order to do this, we need the definition of a droplet, which is just a polygon in \mathbb{Z}^2 . Droplets will be our means of controlling the growth of a set of infected sites.

Definition 2.1. Let $\mathcal{T} \subset S^1$. A \mathcal{T} -droplet is a non-empty set of the form

$$D = \bigcap_{u \in \mathcal{T}} (\mathbb{H}_u + a_u)$$

for some collection $\{a_u \in \mathbb{Z}^2 : u \in \mathcal{T}\}$.

Reinterpreted in terms of droplets, the definition of α in (1) is equivalent to the statement that there exist finite \mathcal{T} -droplets for some set $\mathcal{T} \subset \mathcal{S}$ such that $\alpha(u) \geq \alpha$ for all $u \in \mathcal{T}$, but that the same is not true if α is replaced by any larger quantity. In other words, the closures of finite sets of infected sites are contained in closed droplets such that sets of at least α sites are needed to create non-localized new infections. In the other direction, the condition that there exists an open semicircle $C \subset S^1$ such that every $u \in C$ has difficulty at most α , which is implied by the definition of α , means that there is just large enough an interval of directions all having difficulty at most α for there to exist infinite sequences of nested droplets such that it is possible to grow between consecutive droplets using only sets of α sites. (Note that in the general model, unlike in symmetric bootstrap models, droplets do not necessarily grow in all directions.)

Before continuing with the outline of the proof, let us record two conventions that we use throughout the paper. First, \mathcal{U} is always a (fixed) critical update family, unless explicitly stated otherwise. Using results from Section 2.5, this is equivalent to the statement that $1 \leq \alpha < \infty$. Second, A is always a p -random subset of \mathbb{Z}^2 .

2.1. Upper bounds. The overall approach of the proofs of the upper bounds mirrors that of previous works (see for example [1, 24, 37]). First we obtain a lower bound of the form $\exp(-O(p^{-\alpha}))$ for the probability that a droplet at a particular intermediate scale (which is roughly $p^{-\Theta(1)}$) is (almost) internally filled, where ‘internally filled’ is defined as follows.

Definition 2.2. A set $X \subset \mathbb{Z}^2$ is *internally filled* by A if $X \subset [X \cap A]$. The event that X is internally filled by A is denoted $I(X)$.

In fact, as alluded to before the definition, normally we show that certain droplets are not quite *exactly* internally filled, but *almost* internally filled, where we use this terminology informally to mean that sites within distance $O(1)$ may be used to help

fill the droplet. The resulting loss of independence is not a problem, because the events are increasing and we bound them using Harris's inequality.

The key step in the proof is a bound of the form

$$\mathbb{P}_p\left(D_m \subset [D_{m-1} \cup (D_{m+1} \cap A)]\right) \geq (1 - (1 - p^\alpha)^{\Omega(m)})^{O(1)},$$

where $D_0 \subset D_1 \subset \dots$ is a certain sequence of nested droplets. This bound corresponds to the intuition that it is enough to find somewhere along each side of the droplet a bounded number of sets of α sites contained in A . Once we have this bound, we then deduce that with high probability there exists an internally filled copy of this intermediate droplet within distance \sqrt{t} of the origin, and that with high probability this droplet grows to infect the origin by time t .

All of what we have just said assumes to some extent that the family is balanced. If it is unbalanced then the droplets in the nested sequence $(D_m)_{m=0}^\infty$ are somewhat different: the sides (in the directions of growth) are forced to have constant length, rather than length growing linearly with m (as in the balanced case), and as a consequence the droplets are much less 'regular' (for example, the initial droplet has width λ and height $\lambda p^{-\alpha} \log(1/p)$, where again λ is a large constant, while in the balanced case it has constant size). The growth also features an extra step, in which an extremely long rectangular droplet grows a triangle of infected sites on its side.

Two key deterministic properties of the growth process are needed to make the above ideas work, for both balanced and unbalanced families. The first we have already discussed: the statement that a bounded number of sets of α sites are enough to infect an entire new line; we refer to this principle as 'voracity'. The second is the ability to grow to the corners of droplets, not just to within a bounded distance of the corners. That it is not obvious that this can be done is demonstrated by the fact that in general it *cannot* be done if \mathcal{T} is taken to be a subset of the stable set \mathcal{S} . However, using the idea of 'quasi-stability' introduced in [12], one can show that it can be done if a certain set of unstable directions is included in \mathcal{T} . This is discussed in Section 3.

2.2. The lower bound for balanced families. The lower bound for balanced update families is also not too difficult, but again requires refined versions of arguments from [12]. In order to sketch the proof, let us first briefly recall the argument of Aizenman and Lebowitz [1] for the 2-neighbour model. Their key lemma states that if $A \subset \mathbb{Z}_n^2$ percolates, then for every $1 \leq k \leq n$, there exists an internally filled rectangle of semi-perimeter between k and $2k$. Using the well-known extremal result that such an internally filled rectangle contains at least k initially infected sites, the bound follows from a straightforward calculation.

The key lemma of Aizenman and Lebowitz is proved via the so-called 'rectangles process', which is an algorithm for determining the exact closure of a finite set under the 2-neighbour process. The algorithm proceeds by breaking down the bootstrap process into steps, each of which corresponds to the joining of two nearby rectangles

into a larger rectangle. (Note that rectangles are closed under the 2-neighbour process.) One significant obstacle in the analysis of the general model is the lack of a corresponding *exact* algorithm. Our solution is to use a process analogous to the rectangles process but rather more complicated. This process is an adaptation of the ‘covering algorithm’ of Bollobás, Smith and Uzzell [12], and we use it in order to prove lemmas corresponding to those of [1]. Roughly speaking, we shall treat clusters of α nearby sites as seeds, cover each with a small \mathcal{S} -droplet, and combine them pairwise into larger droplets if they are sufficiently close to interact in the \mathcal{U} -bootstrap process. The crucial deterministic property of the covering algorithm is that the remaining infections (those not in α -clusters) contribute a negligible amount to the set of eventually infected sites; this is proved in Lemma 6.5.

2.3. The lower bound for unbalanced families. The proofs of the previous three parts of the theorem are essentially refinements of established techniques. For this final part of the theorem, however, these techniques do not seem to be useful, and instead we introduce several substantial new ideas, including iterated hierarchies, the u -norm, and icebergs (see below). We mention these ideas only briefly in this section, focusing instead on the broad structure of the proof, and on some of the most important definitions. A much fuller outline of the proof is given at the start of Section 8 (see also Section 6).

The first observation we make (see Lemma 2.7) is that there exist opposite stable directions u^* and $-u^*$ that both have difficulty at least $\alpha + 1$. We set $\mathcal{S}_U = \{u^*, -u^*, u^l, u^r\}$, where u^l and u^r are stable directions on different sides of u^* , each of difficulty at least α , and we consider only \mathcal{S}_U -droplets. Let us rotate our perspective so that u^* is vertical, and write $h(D)$ and $w(D)$ for the height and width of an \mathcal{S}_U -droplet respectively.

As in the balanced case, first we need an approximate rectangles process, which will allow us to say that if a large droplet is internally filled then it must contain droplets at all scales that are approximately internally filled. The covering algorithm is no longer useful to us because it is too crude to capture the biased nature of the geometry of unbalanced models. Instead we use a second algorithm, the ‘spanning algorithm’, which is an adaptation of an idea first used in [15] and subsequently developed in [3, 4, 16, 19]. The algorithm uses the following notion of connectedness.

Definition 2.3. Let κ be a sufficiently large constant, to be defined explicitly in (12). Define a graph G_κ with vertex set \mathbb{Z}^2 and edge set E , where $\{x, y\} \in E$ if and only if $\|x - y\|_2 \leq \kappa$. We say that a set of vertices $S \subset \mathbb{Z}^2$ is *strongly connected* if it is connected in the graph G_κ .

The spanning algorithm allows us to break down the formation of an ‘internally spanned’ droplet into intermediate steps in the same way that the original rectangles process allows us to break down the formation of an internally filled droplet into intermediate steps.

Definition 2.4. Let $\mathcal{T} \subset S^1$. A \mathcal{T} -droplet D is *internally spanned* by A if there is a strongly connected set $L \subset [D \cap A]$ such that D is the smallest \mathcal{T} -droplet containing L . When $\mathcal{T} = \mathcal{S}_U$, the event that D is internally spanned is denoted $I^\times(D)$.

Many previous authors have used the term ‘internally spanned’ to mean (what we refer to as) ‘internally filled’. We emphasize that our terminology (which follows [3, 4], and seems to us more natural) is different.

Using the spanning algorithm we are able to say that if a large droplet is internally spanned, then it contains internally spanned droplets at all smaller scales. The scale we are particularly interested in is the ‘critical’ scale, which for unbalanced models has the following specific meaning.

Definition 2.5. Let \mathcal{U} be unbalanced and let $\xi > 0$ be a small positive constant. An \mathcal{S}_U -droplet D is *critical* if either of the following conditions hold:

- (T) $w(D) \leq p^{-\alpha-1/5}$ and $\frac{\xi}{p^\alpha} \log \frac{1}{p} \leq h(D) \leq \frac{3\xi}{p^\alpha} \log \frac{1}{p}$;
- (L) $p^{-\alpha-1/5} \leq w(D) \leq 3p^{-\alpha-1/5}$ and $h(D) \leq \frac{\xi}{p^\alpha} \log \frac{1}{p}$.

Why might this be the right definition? It is certainly not surprising that the droplet should be long and thin; this is the nature of unbalanced growth, as suggested by the proof of the upper bound in Section 5. The height $h = \frac{\xi}{p^\alpha} \log \frac{1}{p}$ is such that an initial rectangle of height h and constant width will fail to grow sideways (that is, perpendicular to u^*) by a constant distance with probability roughly $p^{O(\xi)}$, and therefore one would expect the rectangle to grow sideways only to distance $p^{-O(\xi)}$. The width $w = p^{-\alpha-1/5}$ is such that the probability the rectangle grows to distance w is sufficiently small to compensate for the number of choices for the initial rectangle. The reason for there being two types of critical droplet is that the spanning algorithm cannot control the width and the height of the critical droplet simultaneously.

In order to bound the probability that a critical droplet D is internally spanned, we shall show that, if the droplet is of type (T), then it is unlikely that $[D \cap A]$ contains a connected set joining the $(-u^*)$ -side of D to the u^* -side, while if it is of type (L), then instead it is unlikely that $[D \cap A]$ contains a strongly connected set joining the u^l -side to the u^r -side. (The u -side of a droplet is defined precisely below.) These events are called ‘vertical crossings’ and ‘horizontal crossings’ respectively.

There are several complications that occur while bounding the probabilities of such crossings. Consider first vertical crossings, and note that, since $\alpha(u^*) \geq \alpha + 1$, we have either $\min \{\alpha^+(u^*), \alpha^-(u^*)\} \geq \alpha + 1$, or

$$\max \{\alpha^+(u^*), \alpha^-(u^*)\} = \infty \quad \text{and} \quad \min \{\alpha^+(u^*), \alpha^-(u^*)\} \geq 1, \quad (2)$$

and similarly for $-u^*$. Since the former case is much easier to handle, let us assume in this sketch that (2) holds. (In this case we say the model ‘exhibits drift’.)

For concreteness, suppose that $\alpha^-(u^*) = \infty$ and $\alpha^+(u^*) = 1$. Since we have a pair $\{u^*, -u^*\}$ of opposite stable directions, we may partition the droplet D into many smaller sub-droplets of the same width, and bound the probability that each

is vertically crossed (possibly with help from above and below) independently, since these events depend on disjoint sets of infected sites. In order to bound these crossing probabilities, we need several new ideas. First, we need a method of controlling the range of the \mathcal{U} -bootstrap process assisted by a half-plane. We achieve this by introducing (in Section 6.3) a third algorithm for approximating the closure of a set of sites, which we call the ‘ u -iceberg algorithm’. Second, we need a close-to-best-possible bound for the probability that certain smaller sub-droplets are internally spanned (following [4], we call these sub-droplets ‘savers’). In order to obtain such a bound, we induct on the size of the droplet being crossed. Finally, we need to deal with the ‘stretched geometry’ of drift models; we do so by introducing a family of norms (the ‘ u -norms’) that compress this geometry until it resembles Euclidean space, and we also introduce a new concept of (‘weak’) connectedness; see Sections 8.2 and 8.3 respectively for the details.

Now consider horizontal crossings, and observe that we no longer have symmetry, since $-u^l$ and $-u^r$ are in general not stable directions. This prevents us from partitioning into sub-droplets as with vertical crossings, and so to overcome this we use the ‘hierarchy method’, which was introduced in [27] and subsequently developed in [3, 4, 20, 25]. We would like to emphasize that the reason for our use of hierarchies is different to that of all previous works: here, the reason is the lack of symmetry between u^l and u^r (which is also why we do not need them for vertical crossings); previously the reason has been to prove sharp thresholds for critical probabilities in symmetric settings.

In order to use hierarchies, we need three further ingredients: a bound on the probability that ‘seeds’ (which are small sub-droplets) are internally spanned; a bound on the probabilities of certain (p times shorter) horizontal crossing events; and a bound on the number of hierarchies with a given number of ‘big seeds’. For these we use the induction hypothesis, and the method described above for vertical crossings (although the details are somewhat simpler in this case). Since our use of induction on the size of the droplet amounts to iterating the above argument α times, we refer to this as the ‘method of iterated hierarchies’.

2.4. Definitions and notation. In this subsection we collect for ease of reference various conventions, definitions and notation that we shall use throughout the paper. First, all constants, including those implied by the notation $O(\cdot)$, $\Omega(\cdot)$ and $\Theta(\cdot)$, are quantities that may depend on \mathcal{U} but not on p . Our asymptotic notation is mostly standard, although we just remark that if f and g are positive real-valued functions of p , then we write $f(p) = \Omega(g(p))$ if $g(p) = O(f(p))$, and we write $f(p) = \Theta(g(p))$ if both $f(p) = O(g(p))$ and $g(p) = O(f(p))$. Furthermore, if c_1 and c_2 are constants, then $c_1 \gg c_2$ means that c_2 is sufficiently small depending on c_1 . (This last piece of notation is non-standard; we trust it will not cause any confusion.)

Moving on to geometric definitions, the unadorned norm $\|\cdot\|$ always denotes the Euclidean norm on \mathbb{R}^2 ; as remarked previously, in Section 6 we define a family of

norms on \mathbb{R}^2 called the ‘ u -norms’, which will always be signified with a subscript u thus: $\|\cdot\|_u$.

Next we give names to certain subsets of the plane. For each $u \in S^1$ and $a \in \mathbb{R}^2$, we define the discrete half-planes

$$\mathbb{H}_u(a) := \{x \in \mathbb{Z}^2 : \langle x - a, u \rangle < 0\},$$

where $\langle \cdot \rangle$ denotes the Euclidean inner product. If $a \in \mathbb{Z}^2$ then we have $\mathbb{H}_u(a) = \mathbb{H}_u + a$, but this is false otherwise. Recall that

$$\ell_u := \{x \in \mathbb{Z}^2 : \langle x, u \rangle = 0\},$$

and that ℓ_u^+ (respectively ℓ_u^-) denotes the subset of ℓ_u consisting of the origin and the sites to the right (respectively left) of the origin as one looks in the direction of u . (Thus $\ell_u^+ \cap \ell_u^- = \{0\}$ and $\ell_u^+ \cup \ell_u^- = \ell_u$.) Observe that if u is rational, meaning that it has rational or infinite slope, then the collection of non-empty discrete lines

$$\left\{ \{x \in \mathbb{Z}^2 : \langle x - a, u \rangle = 0\} : a \in \mathbb{Z}^2 \right\}$$

is a discrete set, naturally indexed by \mathbb{Z} . Thus, set $\ell_u(0) := \ell_u$, and for each $j \in \mathbb{Z}$, let $\ell_u(j)$ denote the j th non-empty discrete line in the direction of u . The final subset of the plane we need a name for is the u -side of a \mathcal{T} -droplet D , where $u \in \mathcal{T}$. This is defined to be the set $\partial(D, u) := D \cap \ell_u(i)$, where i is maximal subject to the set being non-empty.

The previous few definitions will be needed mostly (although not quite exclusively) in Sections 3–5, which deal with the upper bounds of Theorem 1.4. For the lower bounds of the theorem, in Sections 6–8, instead we shall need certain measures of the size of finite subsets of the plane. First, for $u \in S^1$ and a finite set $K \subset \mathbb{Z}^2$, define the u -projection of K ,

$$\pi(K, u) := \max \{ |\langle x - y, u \rangle| : x, y \in K \}. \quad (3)$$

Also, let

$$\text{diam}(K) := \max \{ \pi(K, u) : u \in S^1 \} = \max \{ \|x - y\| : x, y \in K \}$$

be the *diameter* of K . Owing to the biased nature of the geometry, in the unbalanced setting the diameter is usually not a useful measure of the size of K . Instead, we work with the *height*

$$h(K) := \pi(K, u^*) = \max \{ |\langle x - y, u^* \rangle| : x, y \in K \},$$

and the *width*

$$w(K) := \pi(K, u^\perp) = \max \{ |\langle x - y, u^\perp \rangle| : x, y \in K \},$$

where $u^* \in \mathcal{S}$ is as given by Lemma 2.7 below, and $u^\perp \in S^1$ is either of the two unit vectors orthogonal to u^* .

Occasionally we shall want to talk about the distance between a site and a set of sites, or between two sets of sites. We use the following standard conventions:

$$\begin{aligned} \|x - Y\| &:= \min \{ \|x - y\| : y \in Y \}, \\ \text{and} \quad \|X - Y\| &:= \min \{ \|x - y\| : x \in X, y \in Y \}, \end{aligned}$$

whenever X and Y are finite subsets of \mathbb{Z}^2 . We also use analogous conventions for other measures of distance, such as the ‘ u -norms’ and inner products.

We end this subsection with two miscellaneous definitions. The first is a constant ν , which we fix as follows:

$$\nu := \max \left\{ \|x - y\| : x, y \in X \cup \{0\}, X \in \mathcal{U} \right\}. \quad (4)$$

One should think of ν as being one notion of the range of the \mathcal{U} -bootstrap process. (We define another constant ρ , which captures a different aspect of the range of the process, in (11).)

Finally, for $u \in S^1$, let

$$\bar{\alpha}(u) := \min \{ \alpha^+(u), \alpha^-(u) \}. \quad (5)$$

Thus, $\bar{\alpha}(u) = \alpha(u)$ if and only if $\alpha^+(u)$ and $\alpha^-(u)$ are either both finite or both infinite. This notation is needed in order to discuss growth under the action of unbalanced models, particularly models that exhibit drift.

2.5. Basic facts about \mathcal{U} -bootstrap percolation. The \mathcal{U} -bootstrap process exhibits a number of particularly simple and elegant properties, some of which we now recall from [12]. We begin with a description of the stable set \mathcal{S} . We write $[v, w]$ for the closed interval of directions between v and w , and say that $[v, w]$ is *rational* if both v and w have rational or infinite slope relative to \mathbb{Z}^2 .

Lemma 2.6 (Theorem 1.10 of [12]). *The stable set \mathcal{S} is a finite union of rational closed intervals of S^1 . Moreover, if $u \in \mathcal{S}$ then $\alpha(u) < \infty$ if and only if u is an isolated point of \mathcal{S} .*

Strictly speaking the second part of Lemma 2.6 was not proved in [12], but it easily follows from the methods therein. The converse to the first part of Lemma 2.6 is also true (and is part of Theorem 4 of [12]): if $\mathcal{S} \subset S^1$ is any set consisting of a finite union of rational closed intervals, then there exists an update family \mathcal{U} such that $\mathcal{S} = \mathcal{S}(\mathcal{U})$. We shall not use this converse.

We are now in a position to deduce the existence of opposite stable directions u^* and $-u^*$ claimed earlier for unbalanced families \mathcal{U} .

Lemma 2.7. *Let \mathcal{U} be an unbalanced critical update family. Then there exists $u^* \in S^1$ such that*

$$\min \{ \alpha(u^*), \alpha(-u^*) \} \geq \alpha + 1.$$

Proof. By the definition of α , there exists an open semicircle $C \in \mathcal{C}$ such that $\alpha(u) \leq \alpha$ for every $u \in C$. Moreover, since \mathcal{U} is critical we have $\alpha < \infty$. Thus, if one of the endpoints of C has difficulty at most α , then it is an isolated point of \mathcal{S} , by Lemma 2.6. Hence, rotating C slightly, we obtain a closed semicircle C' such that $\alpha(u) \leq \alpha$ for all $u \in C'$. But this contradicts our assumption that \mathcal{U} is unbalanced, hence both endpoints of C have difficulty at least $\alpha + 1$, as required. \square

We shall also use the following property of directions of infinite difficulty. The ‘if’ implications are not explicitly proved in [12], but they are trivial, and we do not use them here.

Lemma 2.8 (Lemma 5.2 of [12]). *Let $[v, w] \subset \mathcal{S}$ with $v \neq w$ be a connected component of \mathcal{S} , and let $u \in [v, w]$. Then $\alpha^-(u) = \infty$ if and only if $u \neq v$ and $\alpha^+(u) = \infty$ if and only if $u \neq w$.*

One final simple but important fact is that if u is not stable then \mathbb{H}_u grows to fill the whole of \mathbb{Z}^2 .

Lemma 2.9 (Lemma 3.1 of [12]). *If $u \notin \mathcal{S}$, then $[\mathbb{H}_u] = \mathbb{Z}^2$.*

Thus for every $u \in S^1$ we have the dichotomy $[\mathbb{H}_u] \in \{\mathbb{H}_u, \mathbb{Z}^2\}$.

2.6. Probabilistic lemmas. We end the section by recalling the correlation inequalities of Harris [26], and van den Berg and Kesten [35]. For definitions of increasing events and disjoint occurrence, and for proofs of both inequalities, see [11].

Lemma 2.10. (Harris’s inequality) *If A and B are increasing events then*

$$\mathbb{P}_p(A \cap B) \geq \mathbb{P}_p(A) \cdot \mathbb{P}_p(B).$$

We write $A \circ B$ for the event that A and B occur disjointly.

Lemma 2.11. (The van den Berg-Kesten inequality) *If A and B are increasing events then*

$$\mathbb{P}_p(A \circ B) \leq \mathbb{P}_p(A) \cdot \mathbb{P}_p(B).$$

We shall apply Harris’s inequality frequently throughout the paper, but the van den Berg-Kesten inequality only once, in Lemma 8.9.

3. VORACITY AND QUASI-STABILITY

In Section 2 we mentioned that there were two important deterministic concepts that we needed in order to make our upper bound proofs work. These were the notions of ‘voracious sets’ and ‘quasi-stable directions’. In this section we introduce and develop these ideas, in preparation for the proofs of the upper bounds of Theorem 1.4 in the two sections to follow.

3.1. Voracity and the infection of new lines. We begin by studying sets of infected sites that are sufficient for stable half-planes to grow.

Definition 3.1. Let $u \in S^1$, and let $Z \subset \mathbb{Z}^2$ be a set of size $|Z| \leq \alpha(u)$. If $[\mathbb{H}_u \cup Z] \cap \ell_u$ is infinite, then we say that Z is *voracious* with respect to u .

It is easy to see that the definition of $\alpha(u)$ implies there exists at least one voracious set for every $u \in \mathcal{S}$. We shall show that a bounded number of voracious sets are sufficient, together with the half-plane \mathbb{H}_u , to infect the line ℓ_u , and moreover that these voracious sets do not need to be near each other. The first step is to show that $[\mathbb{H}_u \cup Z]$ contains a ‘blow-up’ of ℓ_u^+ or ℓ_u^- .

We say that a set $Y \subset \ell_u$ is *semi-periodic* if it contains a homothetic copy of ℓ_u^+ or ℓ_u^- ; that is, if $a + k\ell_u^+ \subset Y$ or $a + k\ell_u^- \subset Y$ for some $a \in \ell_u$ and some $k \in \mathbb{N}$. Note that the definition allows Y to contain other sites as well.

Lemma 3.2. *Let $u \in S^1$ be such that $\alpha(u) \leq \alpha$ and let Z be voracious for u . Then $[\mathbb{H}_u \cup Z] \cap \ell_u$ is semi-periodic.*

Proof. We may assume that $[\mathbb{H}_u \cup Z]$ contains infinitely many sites on the line ℓ_u^+ . We may also assume that u is stable, since otherwise the lemma is trivial. This in particular implies that there exists $a \in \mathbb{Z}^2$ such that $[\mathbb{H}_u \cup Z] \subset \mathbb{H}_u(a)$. Partition $\mathbb{H}_u(a) \setminus \mathbb{H}_u$ into disjoint rectangles $\dots, R_{-1}, R_0, R_1, \dots$, each of width 2ν , with R_{i+1} immediately to the right of R_i for each $i \geq 1$, and set $L_i = R_i \cap [\mathbb{H}_u \cup Z]$.

Now, the condition that $[\mathbb{H}_u \cup Z] \cap \ell_u^+$ is infinite and the definition of ν imply that L_i is not empty for any sufficiently large i . There are only finitely many possible configurations for L_i , so there exist $j \in \mathbb{Z}$ and $r \geq 1$ such that $L_j = L_{j+r}$. Furthermore, the set L_i uniquely determines the set L_{i+1} . It follows that

$$(L_j, \dots, L_{j+r-1}) = (L_{j+r}, \dots, L_{j+2r-1}) = (L_{j+2r}, \dots, L_{j+3r-1}) = \dots,$$

and this is sufficient to prove the lemma. \square

By taking suitable translates of the voracious set Z in Lemma 3.2, it is clear that we can infect the whole of either ℓ_u^+ or ℓ_u^- . In order to return back along the line and infect the rest of ℓ_u , we shall use the following lemma.

Lemma 3.3 (Lemma 32 of [12]). *Let $u \in S^1$. If $\alpha^-(u) < \infty$ then there exists a finite set $Z \subset \ell_u^+$ such that $\ell_u^- \subset [\mathbb{H}_u \cup Z]$.*

We can now easily deduce the following lemma, which says that a bounded number of voracious sets are sufficient (together with \mathbb{H}_u) to infect ℓ_u , and moreover we may choose any suitable translation of each voracious set.

Lemma 3.4. *Let $u \in S^1$ be such that $\alpha(u) \leq \alpha$ and let Z be voracious for u . Then there exist $r \in \mathbb{N}$ and $a_1, \dots, a_r, b \in \ell_u$ such that*

$$\ell_u \subset [\mathbb{H}_u \cup (Z + a_1 + k_1 b) \cup \dots \cup (Z + a_r + k_r b)]$$

for every $k_1, \dots, k_r \in \mathbb{Z}$.

Proof. Let Z be voracious for u and assume that $[\mathbb{H}_u \cup Z]$ contains infinitely many sites on the line ℓ_u^+ . It follows immediately from Lemma 3.2 that there exist $r \in \mathbb{N}$ and $a_1, \dots, a_r, b \in \ell_u$ such that, for every $k_1, \dots, k_r \in \mathbb{Z}$, there exists $a \in \ell_u$ such that $a + \ell_u^+ \subset Y$, where

$$Y := [\mathbb{H}_u \cup (Z + a_1 + k_1 b) \cup \dots \cup (Z + a_r + k_r b)].$$

It remains to show that $a + \ell_u^- \subset Y$. Since $\alpha(u) < \infty$, we must have $\alpha^-(u) < \infty$, so by Lemma 3.3 there exists a finite set $Z' \subset \ell_u^+$ such that $\ell_u^- \subset [\mathbb{H}_u \cup Z']$. But $a + Z' \subset a + \ell_u^+ \subset Y$, so it follows by translating Z' that $a + \ell_u^- \subset Y$, as required. \square

As a consequence of Lemma 3.4, one would expect that a \mathcal{T} -droplet D would ‘grow by one step in direction u ’ with probability at least $(1 - (1 - p^\alpha)^{\Omega(m)})^{O(1)}$, where m is the length of the side of D corresponding to u . This is almost true; however, we have a problem near the corners of D : we may need sites not in D but still below the (extended) u -side of D in order to infect the last $O(1)$ sites. We resolve this problem using another idea from [12]: that of quasi-stable directions.

3.2. Quasi-stability. In many of the simpler bootstrap models, the droplets used as bases for growth are taken with respect to the set of stable directions. Droplets for the 2-neighbour model are rectangles – or, put another way, they are taken with respect to the set $\mathcal{S} = \{e_1, -e_1, e_2, -e_2\}$ of stable directions. Similarly, for symmetric, balanced threshold models, droplets can be taken with respect to the set of stable directions, and the droplets are therefore $2k$ -gons consisting of pairs of parallel sides, for some $k \geq 2$. In this case \mathcal{S} -droplets are suitable bases for growth because, when new infections spread in both directions along each edge of the droplet, the set that results is a new, slightly larger droplet.

The same is not true in general: indeed, as noted above, we may fail to infect some of the sites near the corners of D due to boundary effects. The solution to this problem as used by Bollobás, Smith and Uzzell [12] was to introduce a number of *quasi-stable directions*, which are not stable directions, but which nevertheless are treated as such. Thus, droplets are taken with respect to a certain superset of the stable set. For a comprehensive discussion of quasi-stability, we refer the reader to Section 5.1 of [12].

The next lemma summarizes Definition 33 and Lemma 34 of [12], although we also give a simple self-contained proof. Given a set $\mathcal{T} \subset S^1$, we say that u and v are *consecutive* elements of \mathcal{T} if $u \neq v$ and $\mathcal{T} \cap [u, v] = \{u, v\}$.

Lemma 3.5. *There exists a finite set $\mathcal{Q} \subset S^1$ such that for every pair u, v of consecutive elements of $\mathcal{S} \cup \mathcal{Q}$ there exists an update rule X such that*

$$X \subset (\mathbb{H}_u \cup \ell_u) \cap (\mathbb{H}_v \cup \ell_v).$$

Proof. Form \mathcal{Q} by taking the two unit vectors u and $-u$ perpendicular to x (considered as a vector) for every site $x \in X$ and every update rule $X \in \mathcal{U}$. Formally,

$$\mathcal{Q} := \bigcup_{X \in \mathcal{U}} \bigcup_{x \in X} \{u \in S^1 : \langle u, x \rangle = 0\}.$$

Now suppose u and v are consecutive elements of $\mathcal{S} \cup \mathcal{Q}$ and let $w \in [u, v] \setminus \{u, v\}$. Since w is not stable, there exists an update rule $X \subset \mathbb{H}_w$. Suppose the conclusion of the lemma fails, so

$$X \not\subset (\mathbb{H}_u \cup \ell_u) \cap (\mathbb{H}_v \cup \ell_v).$$

Then without loss of generality there exists $x \in X$ such that $\langle x, v \rangle < 0$ and $\langle x, u \rangle > 0$. But this implies that there exists $w' \in S^1$ perpendicular to x with $w' \in [u, v] \setminus \{u, v\}$, contradicting the construction of \mathcal{Q} . (See Figure 1.) \square

It follows immediately from the lemma that when droplets are taken with respect to $\mathcal{S} \cup \mathcal{Q}$, there are rules that allow the droplets to grow along their sides all the way to the corners: *droplets grow into droplets*.

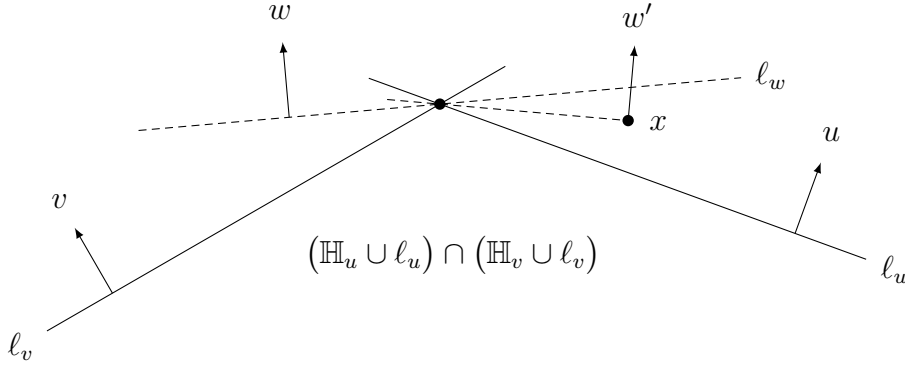


FIGURE 1. Since w is unstable, there exists $X \in \mathcal{U}$ with $X \subset \mathbb{H}_w$. If $x \in X$ lies in the region between ℓ_w and ℓ_u , then the direction w' would be in \mathcal{Q} , by construction, which contradicts u and v being consecutive in $\mathcal{S} \cup \mathcal{Q}$. Thus $X \subset (\mathbb{H}_u \cup \ell_u) \cap (\mathbb{H}_v \cup \ell_v)$, as required.

4. THE UPPER BOUND FOR BALANCED FAMILIES

In this section we shall prove the following theorem, which is the upper bound of Theorem 1.4 for balanced families.

Theorem 4.1. *Let \mathcal{U} be critical and balanced. Then*

$$p^\alpha \log \tau = O(1)$$

with high probability as $p \rightarrow 0$.

Recall that if \mathcal{U} is balanced then there exists a closed semicircle $C \subset S^1$ such that $\alpha(u) \leq \alpha$ for all $u \in C$. Since $\alpha(u) < \infty$ for every $u \in C$, every stable direction $u \in C$ must be isolated, by Lemma 2.6. This implies the existence of a closed arc C' such that $C \subsetneq C' \subsetneq S^1$ and such that $\alpha(u) \leq \alpha$ for all $u \in C'$. Without loss of generality (or more formally, by coupling) we may assume that the endpoints of C' are stable. We call these endpoints u^+ at the left end of C' and u^- at the right end.

Let \mathcal{Q} be the set of quasi-stable directions given by Lemma 3.5 and set

$$\mathcal{S}_Q := (\mathcal{S} \cup \mathcal{Q}) \cap C' \quad \text{and} \quad \mathcal{S}'_Q := \mathcal{S}_Q \setminus \{u^-, u^+\}.$$

Throughout this section droplets will be taken with respect to the set \mathcal{S}_Q .

Choose a collection of vectors $\{a_u \in \mathbb{Z}^2 : u \in \mathcal{S}_Q\}$ and sufficiently large positive constants $\{d_u > 0 : u \in \mathcal{S}'_Q\}$ such that the sequence of \mathcal{S}_Q -droplets

$$D_m := \bigcap_{u \in \{u^-, u^+\}} \mathbb{H}_u(a_u) \cap \bigcap_{u \in \mathcal{S}'_Q} \mathbb{H}_u(a_u + md_u u) \quad (6)$$

for $m = 0, 1, 2, \dots$ have the following properties (see Figure 2):

- (i) D_0 is sufficiently large relative to the d_u ;
- (ii) for every $m \geq 0$ and every consecutive pair $u, v \in \mathcal{S}'_Q$, the intersection⁶ of the lines $\ell_u + a_u + md_u u$ and $\ell_v + a_v + md_v v$ lies on a (continuous) line $L_u^+ = L_v^-$;
- (iii) the lines L_u^+ all intersect at the point $x_0 \in \mathbb{R}^2$, which is also the intersection point of the sides of D_0 corresponding to u^- and u^+ .

Note that we may choose any single one of the d_u arbitrarily (but sufficiently large), independently of D_0 , and then the others will be uniquely determined.

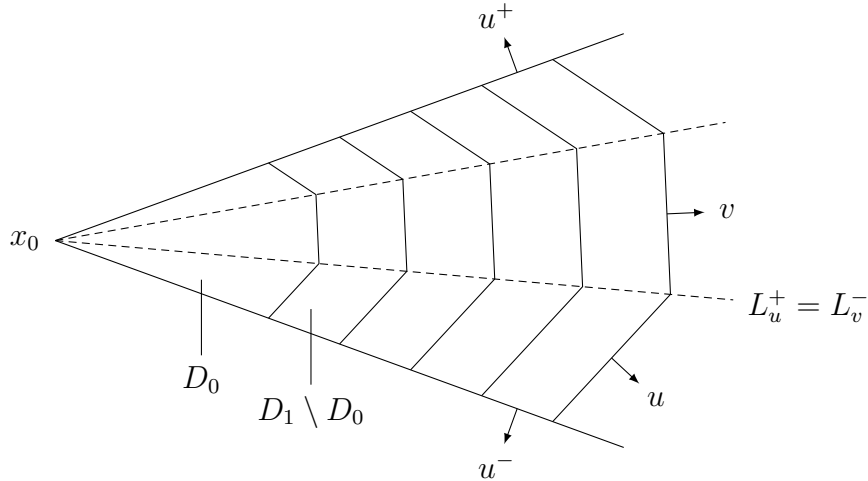


FIGURE 2. The sequence of droplets $D_0 \subset D_1 \subset D_2 \subset \dots$.

⁶These are discrete lines and may have empty intersection. If this is the case then we mean instead the intersection of the corresponding continuous lines; this may not be an element of \mathbb{Z}^2 .

The key lemma in our proof of Theorem 4.1 will be the following bound on the probability that a droplet grows by a constant distance.

Lemma 4.2. *Let $m \in \mathbb{N}$. Then*

$$\mathbb{P}_p(D_m \subset [D_{m-1} \cup (D_{m+1} \cap A)]) \geq (1 - (1 - p^\alpha)^{\Omega(m)})^{O(1)}.$$

Note that the constants implicit in the right-hand side of the inequality above depend on our choice of droplets, and hence on \mathcal{U} , but not on the probability p . Before proving Lemma 4.2, let us show that it implies the following lower bound on the probability that a large droplet is almost internally filled.

Lemma 4.3. *Let $m \in \mathbb{N}$. Then*

$$\mathbb{P}_p(D_m \subset [D_{m+1} \cap A]) \geq \exp(-O(p^{-\alpha})).$$

Proof. Noting that all the events we are considering are increasing, it follows from Harris's inequality (Lemma 2.10) that

$$\begin{aligned} \mathbb{P}_p(D_m \subset [D_{m+1} \cap A]) &\geq \mathbb{P}_p\left(I(D_0) \cap \bigcap_{k=1}^m \left\{D_k \subset [D_{k-1} \cup (D_{k+1} \cap A)]\right\}\right) \\ &\geq \mathbb{P}_p(I(D_0)) \prod_{k=1}^m \mathbb{P}_p(D_k \subset [D_{k-1} \cup (D_{k+1} \cap A)]). \end{aligned}$$

Thus, by Lemma 4.2, we have

$$\begin{aligned} \mathbb{P}_p(D_m \subset [D_{m+1} \cap A]) &\geq p^{O(1)} \prod_{k=1}^{\infty} (1 - (1 - p^\alpha)^{\Omega(k)})^{O(1)} \\ &\geq p^{O(1)} \exp\left(-O(1) \sum_{k=1}^{\infty} -\log(1 - e^{-\Omega(p^\alpha k)})\right) \\ &\geq p^{O(1)} \exp\left(-O(p^{-\alpha}) \int_0^{\infty} -\log(1 - e^{-z}) dz\right) \\ &\geq \exp(-O(p^{-\alpha})), \end{aligned}$$

where for the final inequality we used the fact that $\int_0^{\infty} -\log(1 - e^{-z}) dz < \infty$. \square

From here, the deduction of Theorem 4.1 is straightforward.

Proof of Theorem 4.1. Let λ be a sufficiently large constant, and set

$$p = \left(\frac{\lambda}{\log t}\right)^{1/\alpha}.$$

We shall show that $\tau \leq t$ with high probability as $p \rightarrow 0$.

To avoid some technical issues, let us ‘sprinkle’ the initially infected sites in two rounds; that is, we take $A^{(1)}$ and $A^{(2)}$ to be independent p -random subsets of \mathbb{Z}^2 , and define the set of infected sites to be $A = A^{(1)} \cup A^{(2)}$. Strictly speaking this means we are including sites in A with probability $2p - p^2$, but this is permissible

because we have freedom over the infection probability up to a constant factor. We use the first round of sprinkling to find an almost internally filled copy of D_m , where $m := (\log t)^3$, such that the corresponding copy of $D_{t^{1/3}}$ contains the origin, and the second round to show that the copy of D_m grows (with high probability) to fill the copy of $D_{t^{1/3}}$. This will be enough to prove the theorem, because even if the sites in the copy of $D_{t^{1/3}}$ are infected one-by-one, the total time taken will be $O(t^{2/3}) < t$.

Let us define the following events:

$$E := \bigcup_{x \in \mathbb{Z}^2} \left(\left\{ x + D_m \subset [(x + D_{m+1}) \cap A^{(1)}] \right\} \cap \{ \mathbf{0} \in x + D_{t^{1/3}} \} \right)$$

is the event that $x + D_m$ is ‘almost’ internally filled by $A^{(1)}$, for some $x \in \mathbb{Z}^2$ such that $x + D_{t^{1/3}}$ contains the origin, and

$$F(x) := \left\{ x + D_{t^{1/3}} \subset [(x + D_m) \cup (D_{t^{1/3}+1} \cap A^{(2)})] \right\}$$

is the event that $x + D_{t^{1/3}}$ is ‘almost’ internally filled by $A^{(2)} \cup (x + D_m)$. Since E and $F(x)$ are independent events, and by the comments above, it will suffice to show that $\mathbb{P}_p(E^c) = o(1)$ and $\mathbb{P}_p(F(x)^c) = o(1)$ for an arbitrary vertex $x \in \mathbb{Z}^2$.

To bound $\mathbb{P}_p(E^c)$, observe first that there exists a collection of $\Omega(t^{2/3}/m^2)$ sites $x \in \mathbb{Z}^2$, all satisfying $\mathbf{0} \in x + D_{t^{1/3}}$, such that the sets $x + D_{m+1}$ are pairwise disjoint. By Lemma 4.3, it follows that

$$\mathbb{P}_p(E^c) \leq \left(1 - \exp(-O(p^{-\alpha})) \right)^{\Omega(t^{2/3}/m^2)} \leq \exp\left(-t^{2/3+o(1)} e^{-O(\log t)/\lambda}\right) = o(1)$$

if λ is sufficiently large. Moreover, by Lemma 4.2, for any $x \in \mathbb{Z}^2$,

$$\begin{aligned} \mathbb{P}_p(F(x)) &\geq \prod_{k=m}^{t^{1/3}+1} \left(1 - (1 - p^\alpha)^{\Omega(k)} \right)^{O(1)} \geq \left(1 - (1 - p^\alpha)^{\Omega(m)} \right)^{O(t)} \\ &\geq \exp\left(-O(e^{-\Omega(p^\alpha m)} \cdot t)\right) = 1 - o(1), \end{aligned}$$

as claimed. By the comments above, this completes the proof of the theorem. \square

Our only remaining task is to prove Lemma 4.2. Having already established the deterministic lemmas of the previous section, the idea of the proof is simple. In order to grow from D_m to D_{m+1} it is sufficient for a bounded number of events to occur, each event having failure probability at most $(1 - p^\alpha)^{\Omega(m)}$. These events are all very loosely speaking of the form ‘there exists in A a translate of a given set of α sites somewhere along one of the edges of the droplet’. Since any set of α sites is a subset of A with probability p^α , and since there are $\Omega(m)$ possible disjoint translates of that set, we obtain the desired bound on the probability.

For the sake of completeness, we now present a rigorous proof of Lemma 4.2. Since the proof is easy but notationally rather technical, we encourage the reader who is satisfied with the sketch above to skip ahead to Section 5. We begin by giving a name to the sets that we shall use to grow the droplets.

Definition 4.4. Let $u \in \mathcal{S}'_Q$, let Z be voracious for u , and let $r \in \mathbb{N}$ and $a_1, \dots, a_r, b \in \ell_u$ be given by Lemma 3.4. Let $1 \leq j \leq r$, and for each $i \in \mathbb{Z}$ fix an arbitrary site y_i on the line $\ell_u(i)$. A (u, i, j, α) -cluster is a set of the form

$$Z + y_i + a_j + kb$$

for some $k \in \mathbb{Z}$. We say that a (u, i, j, α) -cluster Z' is *suitable* if it lies between the lines L_u^+ and L_u^- , at a distance at least λ from both, where λ is a sufficiently large constant.

Next, given $u \in \mathcal{S}'_Q$ and $m \in \mathbb{N}$, define

$$I(u, m) = \left\{ i \in \mathbb{Z} : \ell_u(i) \cap D_{m-1} = \emptyset \text{ and } \ell_u(i) \cap D_m \neq \emptyset \right\}. \quad (7)$$

It follows from (6) that $||I(u, m)| - |I(u, m')|| \leq 1$ for every $m, m' \in \mathbb{N}$, but we emphasize that $|I(u, m)|$ and $|I(v, m)|$ can be very different for distinct $u, v \in \mathcal{S}'_Q$. The following lemma states that the event

$$\left\{ D_m \subset [D_{m-1} \cup (D_{m+1} \cap A)] \right\}$$

holds provided that, for each $u \in \mathcal{S}'_Q$, each $i \in I(u, m)$, and each $1 \leq j \leq r = r(u)$, there exists a suitable (u, i, j, α) -cluster contained in A .

Lemma 4.5. *Let $m \in \mathbb{N}$, and for each $u \in \mathcal{S}'_Q$, each $i \in I(u, m)$ and each $1 \leq j \leq r(u)$, let $Z(u, i, j)$ be a suitable (u, i, j, α) -cluster. Then*

$$D_m \subset \left[D_{m-1} \cup \bigcup_{u \in \mathcal{S}'_Q} \bigcup_{i \in I(u, m)} \bigcup_{j=1}^{r(u)} Z(u, i, j) \right].$$

Proof. Given $m \in \mathbb{N}$ and $u \in \mathcal{S}'_Q$, let $i_0 := \min I(u, m)$, and for $v, w \in \mathcal{S}_Q$ such that v, u, w are consecutive in \mathcal{S}_Q define

$$\rho(u, m) := \ell_u(i_0) \cap \mathbb{H}_v(a_v + (m-1)d_v v) \cap \mathbb{H}_w(a_w + (m-1)d_w w),$$

where we set $d_{u-} = d_{u+} = 0$. Thus $\rho(u, m)$ is the set of elements of the line $\ell_u(i_0)$ that lie between the v and w sides of D_{m-1} when those sides are extended, and contains all but a constant number of the elements of $\ell_u(i_0)$ that lie between L_u^- and L_u^+ .

We claim that

$$\rho(u, i_0) \subset \left[D_{m-1} \cup \bigcup_{j=1}^{r(u)} Z(u, i_0, j) \right]. \quad (8)$$

Once we have proved this claim, the lemma will follow, first by iterating this step $|I(u, m)|$ times for each $u \in \mathcal{S}'_Q$, and then by infecting the remaining sites contained in parallelograms at the corners of D_m using Lemma 3.5.

To prove (8), note first that each (u, i_0, j, α) -cluster $Z(u, i_0, j)$ is located sufficiently far from the corners of the droplet, since each is assumed to be suitable.

Therefore, by Lemma 3.4, the right-hand side of (8) contains all sites in $\rho(u, m)$ except possibly a constant number at either end. Now, combining Lemmas 3.3 and 3.5, it follows that these final elements of $\rho(u, m)$ are also infected, as required. \square

Lemma 4.2 is a simple consequence of Lemma 4.5.

Proof of Lemma 4.2. Observe that for every $m \in \mathbb{N}$, $u \in \mathcal{S}'_Q$, $i \in I(u, m)$ and each $1 \leq j \leq r(u)$ there are at least $\Omega(m)$ disjoint suitable (u, i, j, α) -clusters. Since each (u, i, j, α) -cluster is contained in A with probability p^α , it follows by Lemma 4.5 that

$$\mathbb{P}_p(D_m \subset [D_{m-1} \cup (D_{m+1} \cap A)]) \geq (1 - (1 - p^\alpha)^{\Omega(m)})^{O(1)},$$

as required. \square

5. THE UPPER BOUND FOR UNBALANCED FAMILIES

In this final section on upper bounds we prove the following general theorem, which in particular implies the upper bound in Theorem 1.4 for unbalanced families.

Theorem 5.1. *Let \mathcal{U} be a critical update family. Then*

$$p^\alpha \left(\log \frac{1}{p} \right)^{-2} \log \tau = O(1)$$

with high probability as $p \rightarrow 0$.

The theorem does not require the hypothesis that \mathcal{U} is unbalanced, although of course it is only under that assumption that the result is tight up to the implicit constant. It may be helpful in this section to think of \mathcal{U} as being unbalanced, even though this is not strictly necessary.

By the definition of α , there exists an open semicircle $C \subset S^1$ such that $\alpha(u) \leq \alpha$ for all $u \in C$. Let u^\perp be the midpoint of C , let u^* and $-u^*$ be the left and right endpoints of C respectively, and note that $\alpha^+(u^*) < \infty$ (and similarly $\alpha^-(-u^*) < \infty$) by Lemma 2.8. Thus, by Lemma 3.3, there exists a finite set of consecutive sites $Z \subset \ell_{u^*}^+$ such that $\ell_{u^*}^+ \subset [\mathbb{H}_{u^*} \cup Z]$. Define α^* to be the cardinality of Z .

Let \mathcal{Q} be the set of quasi-stable directions given by Lemma 3.5, and set

$$\mathcal{S}_Q := ((\mathcal{S} \cup \mathcal{Q}) \cap C) \cup \{u^*, -u^*, -u^\perp\} \quad \text{and} \quad \mathcal{S}'_Q := (\mathcal{S} \cup \mathcal{Q}) \cap C.$$

In this section all droplets will be \mathcal{S}_Q -droplets. Since the growth process will predominantly take place in directions parallel to the vectors u^\perp and u^* , to simplify the notation we rotate the lattice \mathbb{Z}^2 so that u^* is directed vertically upwards. The discrete rectangle with opposite corners (a, b) and (c, d) is thus defined to be

$$R((a, b), (c, d)) := \{xu^\perp + yu^* \in \mathbb{Z}^2 : x, y \in \mathbb{R}, a \leq x \leq c \text{ and } b \leq y \leq d\}.$$

The sequences of the droplets will be defined in terms of the following quantities:

$$m_1(p) := \frac{\lambda_1}{p^\alpha} \log \frac{1}{p}, \quad m_2(p) := p^{-\lambda_2}, \quad m_3(p) = p^{2\alpha^*} m_2(p) \quad \text{and} \quad m_4(p) := \sqrt{t},$$

where $\lambda_1 \gg \lambda_2 \gg 1$ are sufficiently large positive constants and $t = t(p)$, to be specified later (see (9)), satisfies $\log t \leq p^{-\lambda_2/2}$.

Let

$$\begin{aligned} R_0 &:= R\left((0, 0), (\lambda_1, m_1(p))\right), & R_1 &:= R\left((0, 0), (2m_2(p) + \lambda_1, m_1(p))\right), \\ R_2 &:= R\left((m_2(p), 0), (m_2(p) + \lambda_1, m_1(p) + m_3(p))\right), \\ \text{and } R_3 &:= R\left((m_2(p), 0), (m_4(p), m_1(p) + m_3(p))\right). \end{aligned}$$

be rectangles, and let

$$T := \left\{ xu^\perp + yu^* \in \mathbb{Z}^2 : 0 \leq x \leq m_2(p) + \lambda_1 \text{ and } 0 \leq y - m_1(p) \leq p^{2\alpha^*} x \right\}$$

be a triangle; see Figure 3.

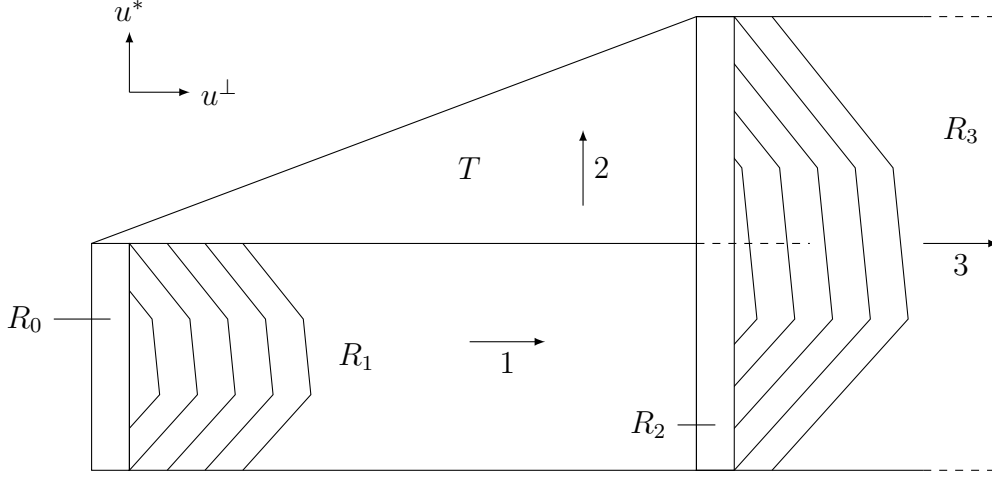


FIGURE 3. The growth mechanism in the unbalanced setting.

For technical reasons, we also need to use the rectangles

$$\begin{aligned} R'_1 &:= R\left((0, 0), (4m_2(p), m_1(p))\right) \\ \text{and } R'_3 &:= R\left((m_2(p), 0), (2m_4(p), m_1(p) + m_3(p))\right), \end{aligned}$$

which are roughly twice as long as R_1 and R_3 respectively.

Figure 3 illustrates the growth mechanism we use to prove Theorem 5.1. It comes in four stages, and, as in the previous section, we use sprinkling to maintain independence between the different stages.

- *Stage 0.* We find a copy of R_0 contained in A such that the corresponding rectangle R_3 contains the origin.
- *Stage 1.* The infection spreads in the direction u^\perp from R_0 and fills the rectangle R_1 . This occurs in a similar way to growth in balanced models, except that the rows are not increasing in size.

- *Stage 2.* The infection spreads in the direction u^* from R_1 using the triangle T to fill R_2 .
- *Stage 3.* Exactly as in Stage 1, the infection spreads in the direction u^\perp from $R_2 \subset T$ to fill R_3 , thus infecting the origin.

In the proof we show that stages 1, 2 and 3 all occur with high probability. The only unlikely part of the growth mechanism is stage 0, the event that $R_0 \subset A$, and for this reason one should think of R_0 as being the ‘critical droplet’.

Our task now is to make the above sketch precise. We postpone the proof of the following key lemma until later in the section.

Lemma 5.2. *The event*

$$\left\{ R_3 \subset [R_0 \cup ((R'_1 \cup T \cup R'_3) \cap A)] \right\}$$

occurs with high probability as $p \rightarrow 0$.

From here, the deduction of Theorem 5.1 is almost immediate.

Proof of Theorem 5.1. The proof is similar to the proof of Theorem 4.1 for balanced families, but the details are simpler. Let $\lambda > 0$ be sufficiently large, and define t by

$$p^\alpha \left(\log \frac{1}{p} \right)^{-2} \log t = \lambda. \quad (9)$$

We shall show that $\tau \leq t$ with high probability as $p \rightarrow 0$.

As before, we sprinkle in two rounds, each round using probability p (which, also as before, is permissible, if a slight abuse of notation), and denote by $A^{(1)}$ and $A^{(2)}$ the sites infected in each step. There exist at least $\Omega(m_4(p))$ choices of $x \in \mathbb{Z}^2$ such that $\mathbf{0} \in x + R_3$ and the sets $x + R_0$ are disjoint, and the probability that $x + R_0 \not\subset A^{(1)}$ for all such x is at most

$$(1 - p^{O(m_1(p))})^{\Omega(m_4(p))} \leq \exp(-t^{1/2 - O(1)/\lambda}) = o(1),$$

since

$$p^{O(m_1(p))} = \exp\left(-\frac{O(1)}{p^\alpha} \left(\log \frac{1}{p}\right)^2\right) = t^{-O(1)/\lambda}$$

and $m_4(p) = \sqrt{t}$.

Now fix x such that $x + R_0 \subset A^{(1)}$ and $\mathbf{0} \in x + D_3$. Rather than continuing to write $x + R_0$, etc., let us modify our notation so that $R_0 \subset A^{(1)}$ and $\mathbf{0} \in R_3$. By Lemma 5.2,

$$\mathbb{P}_p(R_3 \subset [R_0 \cup ((R'_1 \cup T \cup R'_3) \cap A^{(2)})]) = 1 - o(1).$$

Since

$$|R'_1 \cup T \cup R'_3| \leq p^{-O(1)} \sqrt{t} < t,$$

it follows that the origin is infected by time t with high probability (again, even if sites are infected one-by-one), as required. \square

We have reduced our task to that of proving Lemma 5.2. As in the previous section, given the framework of voracity and quasi-stability from Section 3, the idea of the proof is simple. In stage 1 of the process, the probability of advancing a constant number of steps is

$$(1 - (1 - p^\alpha)^{\Omega(m_1(p))})^{O(1)} \leq (1 - p^{\Omega(\lambda_1)})^{O(1)}.$$

Since $\lambda_1 \gg \lambda_2$, the set should grow to fill R_1 , and for similar reasons, the infection spreads out rightwards from R_2 to fill R_3 . Both of these steps are almost the same as the corresponding part of the proof for balanced models. The growth upwards from R_1 through T to fill R_2 is a little different. Since the infection might only spread rightwards when advancing row-by-row in the u^* direction, each set of α^* consecutive infected sites we find when growing upwards through T from R_1 must lie to the right of the previous set. Nevertheless, the probability of filling T (except possibly for a small number of sites near the diagonal) is at least

$$(1 - (1 - p^{\alpha^*})^{\Omega(p^{-2\alpha^*})})^{O(m_3(p))} = 1 - o(1).$$

Once again a rigorous proof requires some notational complexity, and we encourage the reader to skip to the end of the section.

We shall prove the following lemma, which easily implies Lemma 5.2.

Lemma 5.3. *The event*

$$\left\{ R_1 \subset [R_0 \cup (R'_1 \cap A)] \right\} \cap \left\{ R_2 \subset [R_1 \cup (T \cap A)] \right\} \cap \left\{ R_3 \subset [R_2 \cup (R'_3 \cap A)] \right\}$$

occurs with high probability as $p \rightarrow 0$.

It clearly suffices to prove that each of the three events in Lemma 5.3 occurs with high probability; for the first and third this follows almost exactly as in Section 4. We define two sequences of droplets as in (6), except with $u^+ = u^*$ and $u^- = -u^*$ (so the corresponding lines are now parallel), and with $-u^\perp$ added to the set of quasi-stable directions. Specifically, for each $m \in \mathbb{Z}$ and each $i \in \{1, 3\}$, define the \mathcal{S}_Q -droplets

$$D_m^{(i)} := R_i \cap \bigcap_{u \in \mathcal{S}'_Q} \mathbb{H}_u(a_u + md_u u) \quad (10)$$

for some $\{a_u \in \mathbb{Z}^2 : u \in \mathcal{S}'_Q\}$ and sufficiently large positive constants $\{d_u > 0 : u \in \mathcal{S}'_Q\}$ such that:

- $R_{i-1} \subset D_0^{(i)}$;
- for every consecutive pair $u, v \in \mathcal{S}'_Q$, there exists a horizontal line $L_u^+ = L_v^-$ (that is, one parallel to u^\perp) that intersects R_i , such that for every $m \in \mathbb{Z}$, the intersection of $\ell_u + a_u + md_u u$ and $\ell_v + a_v + md_v v$ lies on $L_u^+ = L_v^-$;
- for each $u \in \mathcal{S}'_Q$ and each $m \in \mathbb{N}$, the u -side of $D_m^{(i)}$ has size $\Omega(m_i(p))$.

Note that we shall also need to use $D_m^{(i)}$ for those negative values of m for which the droplet is non-empty, as well as for positive values of m .

The following lemma is essentially Lemma 4.2 applied to the droplets $D_m^{(i)}$, and so the proof is omitted.

Lemma 5.4. *Let $i \in \{1, 3\}$ and $m \in \mathbb{Z}$. Then*

$$\mathbb{P}_p \left(D_m^{(i)} \subset [R_{i-1} \cup D_{m-1}^{(i)} \cup (D_{m+1}^{(i)} \cap A)] \right) \geq (1 - (1 - p^\alpha)^{\Omega(m_i(p))})^{O(1)}. \quad \square$$

Proof of Lemma 5.3. We begin by deducing from Lemma 5.4 that the event

$$\left\{ R_1 \subset [R_0 \cup (R'_1 \cap A)] \right\} \cap \left\{ R_3 \subset [R_2 \cup (R'_3 \cap A)] \right\}$$

occurs with high probability as $p \rightarrow 0$. To see this, let $i \in \{1, 3\}$ and observe that there exists $m \in \mathbb{N}$ such that $R_i \subset D_m^{(i)} \subset D_{m+1}^{(i)} \subset R'_i$, if p is sufficiently small, where $m = O(m_{i+1}(p))$. It follows from Lemma 5.4 that

$$\begin{aligned} \mathbb{P}_p \left(D_m^{(i)} \subset [R_{i-1} \cup (D_{m+1}^{(i)} \cap A)] \right) &\geq \left(1 - (1 - p^\alpha)^{\Omega(m_i(p))} \right)^{O(m_{i+1}(p))} \\ &\geq \exp \left(-O(m_{i+1}(p)) \cdot \exp \left(-\Omega(m_i(p) \cdot p^\alpha) \right) \right) = 1 - o(1) \end{aligned}$$

as $p \rightarrow 0$. Indeed, we have $\exp(\Omega(m_1(p) \cdot p^\alpha)) = p^{-\Omega(\lambda_1)} > p^{-\lambda_2} = m_2(p)$ and

$$\exp(\Omega(m_3(p) \cdot p^\alpha)) = \exp(\Omega(p^{-\lambda_2+2\alpha^*+\alpha})) > \sqrt{t} = m_4(p),$$

where we used our assumptions that $\lambda_1 \gg \lambda_2 \gg 1$ and $\log t \leq p^{-\lambda_2/2}$.

It remains to show that the event

$$\left\{ R_2 \subset [R_1 \cup (T \cap A)] \right\}$$

occurs with high probability as $p \rightarrow 0$. To do so, consider the set U_i of the leftmost $p^{-2\alpha^*}$ sites of $T \cap \ell_{u^*}(i)$ for each line $\ell_{u^*}(i)$ that intersects T . Now, suppose that, for every such line, the middle $p^{-2\alpha^*}/3$ sites of U_i contain a set of α^* consecutive sites of A . Then $R_2 \subset [R_1 \cup (T \cap A)]$, by the definition of α^* . But this has probability at least

$$(1 - (1 - p^{\alpha^*})^{p^{-2\alpha^*}/3\alpha^*})^{m_3(p)} \geq \exp \left(-p^{-\lambda_2} \exp(-p^{-\alpha^*/2}) \right) = 1 - o(1),$$

as required. \square

6. APPROXIMATELY INTERNALLY FILLED SETS

In this section we lay the groundwork for the proof of the lower bounds by defining and proving basic properties of three of our key tools: the *covering*, *spanning* and *iceberg* algorithms. These should all be thought of as ways of using droplets to approximate the closure of A under the \mathcal{U} -bootstrap process.

The covering algorithm replaces the rectangles process in the balanced case, and allows us to find inside the component that infects the origin a small region containing many strongly connected subsets of A of cardinality α . For unbalanced models, we use the spanning algorithm to find an internally spanned critical droplet and to

construct an iterated sequence of ‘hierarchies’ for this droplet, and we use the iceberg algorithm to bound the range of the \mathcal{U} -bootstrap process in certain directions with the help of half-planes.

Having completed the proofs of the upper bounds of Theorem 1.4, we no longer have any need for quasi-stable directions, nor indeed for stable directions with difficulty less than α . In fact, henceforth all droplets will be assumed to be taken with respect to a specific finite set of stable directions, which we now define, and whose properties differ according to whether \mathcal{U} is balanced or unbalanced.

If \mathcal{U} is balanced then the set is denoted \mathcal{S}_B and is chosen such that:

- (i) $\bar{\alpha}(u) \geq \alpha$ for every $u \in \mathcal{S}_B$; and
- (ii) \mathcal{S}_B intersects every open semicircle in the unit circle⁷.

It is easy to verify that these conditions can be satisfied, by the definition of α and by the characterization of stable sets in Lemma 2.6. Note that we may assume $|\mathcal{S}_B| \leq 4$ if we wish, although we shall not need to do so.

If instead \mathcal{U} is unbalanced, then the set is denoted \mathcal{S}_U and is such that:

- (i) $\mathcal{S}_U = \{u^*, -u^*, u^l, u^r\}$ for some $u^*, u^l, u^r \in S^1$ such that u^l lies in the open semicircle to the left of u^* and u^r in the open semicircle to the right;
- (ii) $\min\{\alpha(u^*), \alpha(-u^*)\} \geq \alpha + 1$; and
- (iii) $\min\{\bar{\alpha}(u^l), \bar{\alpha}(u^r)\} = \alpha$.

It is again easy to verify, by Definitions 1.2 and 1.3 and Lemmas 2.6 and 2.7, that such a set \mathcal{S}_U exists.

In Section 2.4 we defined ν to be the diameter of \mathcal{U} ,

$$\nu = \max\{\|x - y\| : x, y \in X \cup \{0\}, X \in \mathcal{U}\},$$

and we stated that this would be one measure we would use of the range of the \mathcal{U} -bootstrap process. For balanced models, we shall need the following additional measure of the range of the process:

$$\rho := \max\{\|y - Z\| : |Z| = \alpha - 1, y \in [\mathbb{H}_u \cup Z] \setminus \mathbb{H}_u, u \in \mathcal{S}_B\}, \quad (11)$$

where the maximum is taken over all $u \in \mathcal{S}_B$ and over all choices of $Z \subset \mathbb{Z}^2$ with $|Z| = \alpha - 1$. Note that ρ is finite; this is because \mathcal{S}_B is finite, and because $[\mathbb{H}_u \cup Z] \setminus \mathbb{H}_u$ is finite for all $u \in \mathcal{S}_B$, since $\bar{\alpha}(u) \geq \alpha$ for each such u .

The constant κ in Definition 2.3 of a strongly connected set will need to be quite a bit larger than ν and, for balanced models, quite a bit larger than ρ , so let us set

$$\kappa = \kappa(\mathcal{U}) := \begin{cases} 2(\rho + \nu) & \text{if } \mathcal{U} \text{ is balanced, and} \\ 3\nu & \text{if } \mathcal{U} \text{ is unbalanced.} \end{cases} \quad (12)$$

Recall that sites x and y are said to be strongly connected if $\|x - y\| \leq \kappa$.

⁷Equivalently, the origin lies in the interior of the convex hull of \mathcal{S}_B . Also equivalently, \mathcal{S}_B -droplets are finite.

Finally, given a finite set $K \subset \mathbb{Z}^2$, observe that there is a unique minimal \mathcal{S}_B -droplet $D(K)$ containing K .

6.1. The covering algorithm: balanced families. Throughout this subsection we assume that \mathcal{U} is balanced and that droplets are taken with respect to \mathcal{S}_B . We define the collection of α -covers of a finite set K , and use this to prove two key lemmas: an ‘Aizenman-Lebowitz lemma’, which says that an α -covered droplet contains α -covered droplets of all intermediate sizes, and an extremal lemma, which says that an α -covered droplet contains many disjoint ‘ α -clusters’. The proofs of both lemmas are straightforward applications of the covering algorithm.

The key complication arising from the algorithm is that an α -cover of a set K does not necessarily contain the closure of K under the \mathcal{U} -bootstrap process. However, an approximate version of this statement is true, and this is proved in Lemma 6.5. Roughly speaking, the lemma says that one can obtain (a superset of) the closure $[K]$ from an α -cover of K via only ‘local’ modifications.

We define an α -cluster to be any strongly connected set of α sites. These will be our basic building blocks in the covering algorithm.

Definition 6.1. (*The α -covering algorithm.*) Let \mathcal{U} be balanced. Suppose that we are given:

- K , a finite set of infected sites;
- B_1, \dots, B_{k_0} , a maximal collection⁸ of disjoint α -clusters in K ;
- $\mathcal{D}^0 = \{D_1^0, \dots, D_{k_0}^0\}$, a collection of copies of a fixed, sufficiently large \mathcal{S}_B -droplet \hat{D} , such that $B_j \subset D_j^0$ for each $j = 1, \dots, k_0$.

Set $t := 0$ and repeat the following steps until STOP:

1. If there are two droplets $D_i^t, D_j^t \in \mathcal{D}^t$ and an $x \in \mathbb{Z}^2$ such that the set

$$D_i^t \cup D_j^t \cup (x + \hat{D}) \tag{13}$$

is strongly connected, then set

$$\mathcal{D}^{t+1} := (\mathcal{D}^t \setminus \{D_i^t, D_j^t\}) \cup \{D(D_i^t \cup D_j^t)\},$$

and set $t := t + 1$.

2. Otherwise set $T := t$ and STOP.

The output of the algorithm is the family $\mathcal{D} := \{D_1^T, \dots, D_k^T\}$, where $k = k_0 - T$.

Thus, at each step of the algorithm, we take two nearby droplets in our collection, and replace them by the smallest droplet containing their union. Let us fix from now on a sufficiently large \mathcal{S}_B -droplet \hat{D} as in the covering algorithm.

Definition 6.2. We say that $\mathcal{D} = \{D_1, \dots, D_k\}$ is an α -cover of a finite set $K \subset \mathbb{Z}^2$ if \mathcal{D} is a possible output of the α -covering algorithm with input K . We say that a droplet D is α -covered if the single droplet $\mathcal{D} = \{D\}$ is an α -cover of $D \cap A$.

⁸This collection is of course not uniquely defined; each such collection gives rise to a (possibly different) α -cover of K .

The first important property of the α -covering algorithm is given by the following lemma. We call this result an ‘Aizenman-Lebowitz lemma for α -covered droplets’, since the corresponding result for the 2-neighbour process was first proved in [1]. Let λ be a sufficiently large constant, depending on \hat{D} .

Lemma 6.3. *Let D be an α -covered droplet. Then for every $\lambda \leq k \leq \text{diam}(D)$ there exists an α -covered droplet $D' \subset D$ such that $k \leq \text{diam}(D') \leq 3k$.*

Proof. The lemma is an immediate consequence of two simple observations: that the droplets $D_i^t \in \mathcal{D}^t$ obtained during the α -covering algorithm are all α -covered, and that at each step of the algorithm,

$$\max \{ \text{diam}(D_i^t) : D_i^t \in \mathcal{D}^t \}$$

at most triples in size, provided that this maximum is at least an absolute constant.

To prove the first observation, simply run the algorithm on $D_i^t \cap A$, using the same α -clusters. To prove the second, observe that if droplets D_i^t and D_j^t are united in step t of the algorithm, then by definition there exists $x \in \mathbb{Z}^2$ such that the distance between D_i^t and $x + \hat{D}$, and that between D_j^t and $x + \hat{D}$, are at most κ . Since for any two intersecting droplets D_1 and D_2 we have the easy geometric inequality

$$\text{diam}(D(D_1 \cup D_2)) \leq \text{diam}(D_1) + \text{diam}(D_2),$$

it follows that

$$\text{diam}(D(D_i^t \cup D_j^t)) \leq \text{diam}(D_i^t) + \text{diam}(D_j^t) + O(1), \quad (14)$$

completing the proof of the observation, and hence the lemma. \square

The algorithm also admits the following extremal result, which says that the number of initial α -clusters in an α -covered droplet must be at least linear in the diameter of the droplet. It is precisely because of the existence of this result that we use the α -covering algorithm in the balanced setting, rather than the spanning algorithm defined below, for which there is no correspondingly strong extremal lemma.

Lemma 6.4 (Extremal lemma for α -covered droplets). *Let D be an α -covered droplet. Then $D \cap A$ contains $\Omega(\text{diam}(D))$ disjoint α -clusters.*

Proof. The algorithm begins with k_0 disjoint α -clusters, and ends with $\mathcal{D} = \{D\}$. At each step of the algorithm the number of droplets is reduced by 1, and the sum of the diameters of the droplets increases by at most a constant, by (14). Hence

$$\text{diam}(D) \leq k_0 \text{diam}(\hat{D}) + O(k_0),$$

and so $k_0 = \Omega(\text{diam}(D))$, as required. \square

It remains to show that an α -cover \mathcal{D} of a set K is a reasonable approximation of the closure $[K]$. Lemma 6.5 makes this notion precise, and will be a crucial tool in the application of the α -covering algorithm to balanced update families. The basic idea is simple: since all α -clusters are contained in some droplet of \mathcal{D} , the remaining

‘dust’ of $K \setminus (D_1 \cup \dots \cup D_k)$ should contribute only locally to the set of eventually infected sites.

We remark that a simplified version of the covering algorithm was used in [12], not requiring Lemma 6.5, and in most cases resulting in non-optimal bounds.

Lemma 6.5. *Let \mathcal{U} be balanced. Let $K \subset \mathbb{Z}^2$ be a finite set, let $\mathcal{D} = \{D_1, \dots, D_k\}$ be an α -cover of K , and set $Y := K \setminus (D_1 \cup \dots \cup D_k)$. Then $\|x - Y\| \leq \rho$ for all $x \in [K] \setminus (D_1 \cup \dots \cup D_k)$.*

Before proving the lemma, we need a straightforward statement that allows us to use \mathcal{S}_B -droplets in place of half-planes in the definition (11) of ρ .

Lemma 6.6. *Let \mathcal{U} be balanced. Let D be an \mathcal{S}_B -droplet and let $Y \subset \mathbb{Z}^2$ have size at most $\alpha - 1$. Then $\|x - Y\| \leq \rho$ for all $x \in [D \cup Y] \setminus D$.*

Proof. Let $\{a_u \in \mathbb{Z}^2 : u \in \mathcal{S}_B\}$ be a collection of vectors such that

$$D = \bigcap_{u \in \mathcal{S}_B} \mathbb{H}_u(a_u),$$

and let $x \in [D \cup Y] \setminus D$. Since $x \notin D$, there exists $u \in \mathcal{S}_B$ such that $x \notin \mathbb{H}_u(a_u)$. But $|Y \setminus \mathbb{H}_u(a_u)| \leq \alpha - 1$, and

$$x \in [\mathbb{H}_u(a_u) \cup Y] \setminus \mathbb{H}_u(a_u),$$

so $\|x - Y\| \leq \rho$ by the definition of ρ . \square

Proof of Lemma 6.5. We prove a slightly stronger statement: setting

$$X = \bigcup_{D \in \mathcal{D}} D \quad \text{and} \quad Z = [X \cup Y],$$

we shall show that the same conclusion holds with $[K]$ replaced by Z .

To begin, we partition Y into a collection Y_1, \dots, Y_s of maximal strongly connected components, so in particular if $y \in Y_i$ and $z \in Y_j$ for some $i \neq j$, then $\|y - z\| > 2(\rho + \nu)$. (Note that the sets Y_i are uniquely defined.) By the definition of an α -cover, we must have $|Y_i| \leq \alpha - 1$ for every $i \in [s]$. For each $0 \leq i \leq s$, set

$$Z_i := [X \cup Y_1 \cup \dots \cup Y_i] \setminus X.$$

We claim that, for each $1 \leq i \leq s$,

- (i) $\|x - Y_i\| \leq \rho$ for every $x \in Z_i \setminus Z_{i-1}$, and
- (ii) $\|x - y\| > \rho + 2\nu$ for every $x \in Z_i$ and $y \in Y_{i+1} \cup \dots \cup Y_s$.

We shall prove the claim by induction on i . The lemma will then follow from (i).

When $i = 0$, we have $Z_i = \emptyset$, and both parts of the claim trivially hold. Suppose the claim holds for $i - 1$ and consider Y_i . There is at most one droplet $D \in \mathcal{D}$ such that $Y_i \cup D$ is strongly connected, since \mathcal{D} was the output of the α -covering algorithm, and since \hat{D} was chosen to be sufficiently large. Therefore, setting

$$Y'_i := [X \cup Y_i] \setminus X,$$

we have $(*) \|y - Y_i\| \leq \rho$ for every $y \in Y'_i$, by Lemma 6.6.

Next we shall show that $X \cup Z_{i-1} \cup Y'_i$ is closed under the \mathcal{U} -bootstrap process. To see this, note first that both $X \cup Y'_i$ and $X \cup Z_{i-1}$ are closed, by definition, and therefore if $X \cup Z_{i-1} \cup Y'_i$ is not closed then there must be an update rule intersecting both Z_{i-1} and Y'_i . But, by part (ii) of the induction hypothesis and the previous observation, we have

$$\|Y'_i - Z_{i-1}\| \geq \|x - Y_i\| - \|y - Y_i\| > 2\nu$$

for some $x \in Z_{i-1}$ and $y \in Y'_i$, so this is impossible. Now, since

$$Z_i = [X \cup Z_{i-1} \cup Y_i] \setminus X = [X \cup Z_{i-1} \cup Y'_i] \setminus X,$$

it follows that $Z_i = Z_{i-1} \cup Y'_i$. Part (i) of the claim now follows from this and $(*)$.

For part (ii), let $x \in Z_i$, and observe that if $x \in Z_{i-1}$, then (ii) follows from the induction hypothesis. Therefore assume that $x \in Z_i \setminus Z_{i-1} \subset Y'_i$, and note that hence, by $(*)$, there exists $y \in Y_i$ such that $\|x - y\| \leq \rho$. Let $z \in Y_{i+1} \cup \dots \cup Y_k$, and observe that

$$\|x - z\| \geq \|y - z\| - \|x - y\| > \rho + 2\nu,$$

as claimed, since Y_1, \dots, Y_k are maximal strongly connected components of Y , and $\kappa \geq 2(\rho + \nu)$. This completes part (ii) of the claim, and thus also the lemma. \square

6.2. The spanning algorithm: unbalanced families. Next we describe our second analogue of the rectangles process, which will be a key tool in our analysis of unbalanced models. Throughout this subsection we assume that \mathcal{U} is unbalanced and that droplets are taken with respect to \mathcal{S}_U . This is not strictly speaking necessary: unlike in the previous subsection, the results here hold for \mathcal{T} -droplets for any $\mathcal{T} \subset \mathcal{S}$ such that \mathcal{T} -droplets are finite. Nevertheless, the only applications of the results in this subsection will be to unbalanced families, and it is useful to have the set \mathcal{T} fixed.

Recall from Section 2 that an \mathcal{S}_U -droplet D is said to be internally spanned by A if there exists a strongly connected set $L \subset [D \cap A]$ such that $D(L) = D$. (In this subsection, $D(L)$ denotes the smallest \mathcal{S}_U -droplet containing L .) Given a finite set K of infected sites, the output of the spanning algorithm is a minimal collection \mathcal{D} of internally spanned \mathcal{S}_U -droplets whose union contains K . At each step of the algorithm we maintain a partition $\mathcal{K}^t = \{K_1^t, \dots, K_k^t\}$ of K such that each set $[K_j^t]$ is strongly connected.

Definition 6.7. (*The spanning algorithm.*) Let $K = \{x_1, \dots, x_{k_0}\}$ be a finite set of infected sites. Set $\mathcal{K}^0 := \{K_1^0, \dots, K_{k_0}^0\}$, where $K_j^0 := \{x_j\}$ for each $1 \leq j \leq k_0$. Set $t := 0$, and repeat the following steps until STOP:

1. If there are two sets $K_i^t, K_j^t \in \mathcal{K}^t$ such that the set

$$[K_i^t] \cup [K_j^t] \tag{15}$$

is strongly connected, then set

$$\mathcal{K}^{t+1} := (\mathcal{K}^t \setminus \{K_i^t, K_j^t\}) \cup \{K_i^t \cup K_j^t\},$$

and set $t := t + 1$.

2. Otherwise set $T := t$ and STOP.

The output of the algorithm is the *span* of K ,

$$\langle K \rangle := \{D([K_1^T]), \dots, D([K_k^T])\},$$

where $k = k_0 - T$.

The following lemma shows that we can use the spanning algorithm to determine whether or not D is internally spanned.

Lemma 6.8. *An \mathcal{S}_U -droplet D is internally spanned if and only if $D \in \langle D \cap A \rangle$.*

Proof. We claim that, for every finite set K , we have

$$\langle K \rangle = \{D(L_1), \dots, D(L_k)\}, \quad (16)$$

where L_1, \dots, L_k are the strongly connected components of $[K]$. Applying this to $K = D \cap A$, we see that $D \in \langle D \cap A \rangle$ if and only if $D(L) = D$ for some strongly connected component L of $[D \cap A]$. But $[D \cap A] \subset D$, since $\mathcal{S}_U \subset \mathcal{S}$, and so this is equivalent to the event that D is internally spanned.

To prove the claim, we shall show that the sets $[K_i^T]$ are precisely the strongly connected components of $[K]$. Indeed, it follows from (15) (and a simple induction on t) that $[K_i^t]$ is strongly connected for every $t \in [T]$ and $1 \leq i \leq k_0 - t$, and no two sets $[K_i^T]$ and $[K_j^T]$ are strongly connected, since the algorithm stopped at step T . Moreover, $[K] = \bigcup_{i=1}^k [K_i^T]$, since $\kappa > \nu$, and so no site can be infected by two or more of these sets. \square

We can now prove the ‘Aizenman-Lebowitz lemma for internally spanned droplets’, which is the spanning analogue of Lemma 6.3 for α -covered droplets. For the applications we shall need a slightly more general statement than before. Recall that $\pi(D, u)$ denotes the size of the projection of D in the direction u , and that λ is a sufficiently large constant.

Lemma 6.9. *Let D be an internally spanned \mathcal{S}_U -droplet, and let $u \in S^1$. Then for every $\lambda \leq k \leq \pi(D, u)$, there exists an internally spanned \mathcal{S}_U -droplet $D' \subset D$ with $k \leq \pi(D', u) \leq 3k$.*

Proof. Apply the spanning algorithm to $K = D \cap A$ and observe that, for every $t \leq T$ and every $1 \leq i \leq k_0 - t$, the droplet $D([K_i^t])$ is internally spanned, since $K_i^t \subset D([K_i^t]) \cap A$ and $[K_i^t]$ is strongly connected.

We claim that

$$\max \{ \pi(D([K_i^t]), u) : K_i^t \in \mathcal{K}^t \}$$

at most triples in size at each step, provided that this maximum is at least an absolute constant. To see this, simply note that

$$\pi(D(D_1 \cup D_2), u) \leq \pi(D_1, u) + \pi(D_2, u) + O(1), \quad (17)$$

for any pair of droplets D_1 and D_2 that are within distance $O(1)$ of one another, and that $D([Y]) = D(Y)$ for any set Y , since $\mathcal{S}_U \subset \mathcal{S}$. The lemma now follows easily, as in the proof of Lemma 6.3. \square

We can now deduce an extremal lemma which, while much weaker than the corresponding lemma for α -covered droplets (Lemma 6.4), is in fact tight up to the implicit constant. This fact underlines how much we are ‘giving away’ in assuming only that our droplets are spanned (rather than filled). Nevertheless, this lemma will be sufficient to prove the base case (Lemma 6.11 below) of the main induction argument (Lemma 8.4) for unbalanced models in Section 8.

Lemma 6.10. (Extremal lemma for internally spanned droplets.) *Let D be an internally spanned \mathcal{S}_U -droplet. Then $|D \cap A| = \Omega(\text{diam}(D))$.*

Proof. As in the proof of the previous lemma, we apply the spanning algorithm with $K = D \cap A$. The algorithm starts with k_0 sets containing the individual elements of $D \cap A$, and it finishes with a collection

$$\langle D \cap A \rangle = \{D([K_1^T]), \dots, D([K_k^T])\}$$

such that $D \in \langle D \cap A \rangle$. At each step of the algorithm the number of sets in the collection decreases by 1, and the sum of the diameters of the minimal droplets containing those sets increases by at most a constant, by (17). Hence,

$$\text{diam}(D) \leq \sum_{i=1}^k \text{diam}(D([K_i^T])) \leq k_0 \text{diam}(D([K_1^0])) + O(k_0) = O(k_0),$$

which implies that $k_0 = \Omega(\text{diam}(D))$, as required. \square

Using Lemma 6.10, we can deduce a non-trivial bound on the probability that a very small droplet is internally spanned. As noted before, this will form the base case of our induction argument in Lemma 8.4.

Recall that $I^\times(D)$ denotes the event that the \mathcal{S}_U -droplet D is internally spanned.

Lemma 6.11. *For every $\eta > 0$, there exists $\delta > 0$ such that the following holds. Let D be an \mathcal{S}_U -droplet such that*

$$\lambda \leq \min\{w(D), h(D)\} \leq p^{-1+\eta}.$$

Then

$$\mathbb{P}_p(I^\times(D)) \leq p^{\delta \max\{w(D), h(D)\}}.$$

Proof. Let us write $m(D) := \min \{w(D), h(D)\}$ and $M(D) := \max \{w(D), h(D)\}$. Suppose the \mathcal{S}_U -droplet D is internally spanned. Then by Lemma 6.10, $D \cap A$ must contain at least $\Omega(M(D))$ sites. The probability that this occurs is at most

$$\binom{O(w(D) \cdot h(D))}{\delta' \cdot M(D)} p^{\delta' M(D)} \leq (O(1) \cdot m(D) \cdot p)^{\delta' M(D)} \leq p^{\delta M(D)},$$

for some $\delta, \delta' > 0$, as required. \square

The final lemma of this subsection will be used in Section 8.1 as part of an induction argument to prove the existence of ‘good and satisfied hierarchies’ for internally spanned droplets; see the definitions and Lemma 8.8 contained within that section for details.

Lemma 6.12. *Let $K \subset \mathbb{Z}^2$, with $2 \leq |K| < \infty$, be such that $[K]$ is strongly connected. Then there exists a partition $K = K_1 \cup K_2$ into non-empty (disjoint) sets such that $[K_1]$, $[K_2]$ and $[K_1] \cup [K_2]$ are all strongly connected.*

Proof. Run the spanning algorithm on K and consider the penultimate step. Since $[K]$ is strongly connected, and therefore $\langle K \rangle = \{D([K])\}$, we have

$$\mathcal{K}^{T-1} = \{K_1, K_2\}$$

for some $K_1 \subsetneq K$ and $K_2 \subsetneq K$ such that $K = K_1 \cup K_2$. By their construction in the spanning algorithm, both $[K_1]$ and $[K_2]$ are strongly connected, and since K_1 and K_2 combine at the final step, so too is $[K_1] \cup [K_2]$. \square

6.3. The iceberg algorithm: unbalanced families with drift. Our third algorithm will play a crucial role in the proof for update families that exhibit drift. Assume that \mathcal{U} is unbalanced and let $\{u^*, -u^*\}$ be the pair of stable directions given by Lemma 2.7, so in particular

$$\min \{\alpha(u^*), \alpha(-u^*)\} \geq \alpha + 1.$$

Recall from Section 2.3 that \mathcal{U} exhibits drift if not all of $\alpha^+(u^*)$, $\alpha^-(u^*)$, $\alpha^+(-u^*)$ and $\alpha^-(-u^*)$ are finite. Let us assume⁹ that $\alpha^-(u^*) = \infty$, and observe that $1 \leq \alpha^+(u^*) < \infty$.

When our droplet is growing in direction u^* in a model with drift, it will tend to form a triangle, as in Section 5. In order to control the growth in this direction, we therefore need to ‘give away’ this triangle (in fact, a slightly larger one), and bound the growth outside it. The point of the algorithm defined in this subsection is exactly to control this outwards growth using ‘icebergs’, defined as follows.

Since $\alpha^-(u^*) = \infty$, there exists a non-trivial interval $[u_0, u^*]$ such that $\alpha^-(u) = \infty$ for every $u \in [u_0, u^*]$. Fix such a u_0 , and moreover let us assume that u_0 is chosen sufficiently close to u^* , in a sense made precise below (see Lemma 6.14).

⁹If \mathcal{U} does not exhibit drift then we shall not need the results proved in this section.

Definition 6.13. Let $u \in [u_0, u^*]$. A u -iceberg is any non-empty set J of the form

$$J = (\mathbb{H}_{u_0}(a) \cap \mathbb{H}_{u^*}(b)) \setminus \mathbb{H}_u,$$

where $a, b \in \mathbb{Z}^2$. If X is a finite set of sites such that $X \not\subset \mathbb{H}_u$, then denote by $J_u(X)$ the smallest u -iceberg such that $X \subset \mathbb{H}_u \cup J_u(X)$.

Thus a u -iceberg is a discrete triangle whose sides are perpendicular to $-u$, u^* and u_0 ; see Figure 4. We make a simple but key observation.

Lemma 6.14. *If J is a u -iceberg, then $\mathbb{H}_u \cup J$ is closed.*

Proof. The lemma follows from our choice of u_0 , in particular from the fact that u_0 is sufficiently close to u^* . Indeed, we may choose u_0 closer to u^* than any $v \in S^1 \setminus \{u^*\}$ perpendicular to $x - y$, where $x, y \in \bigcup_{X \in \mathcal{U}} X \cup \{0\}$ and $x \neq y$. This follows easily from the fact that \mathcal{U} is a finite collection of finite sets.

Now, suppose that there exists $z \notin \mathbb{H}_u \cup J$ and a rule $X \in \mathcal{U}$ such that $z + X \subset \mathbb{H}_u \cup J$. Since $u, u_0, u^* \in \mathcal{S}$, there must exist $x, y \in z + X$ with $x \notin \mathbb{H}_u$ and $y \notin \mathbb{H}_{u_0}(a) \cap \mathbb{H}_{u^*}(b)$. But now $x - y$ is perpendicular to a vector in the interval (u_0, u^*) , which contradicts our choice of u_0 . \square

Observe that the construction of u_0 in the previous proof is similar to that of the set \mathcal{Q} in Lemma 3.5; see also Figure 1.

We are now ready to introduce the iceberg algorithm, which is a modified version of the covering algorithm that allows sites to be infected with the help of \mathbb{H}_u .

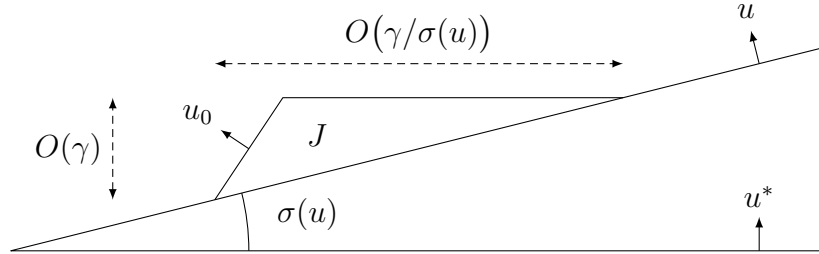


FIGURE 4. A u -iceberg J .

Let us fix a sufficiently large \mathcal{S}_U -droplet \hat{D}_U .

Definition 6.15. (*The u -iceberg algorithm.*) Let \mathcal{U} be an unbalanced update family that exhibits drift, and let u^* and u_0 be as defined above. Suppose we are given:

- $K = \{x_1, \dots, x_{k_0}\} \subset \mathbb{Z}^2 \setminus \mathbb{H}_u$, a finite set of infected sites;
- $\mathcal{W}^0 = \{W_1^0, \dots, W_{k_0}^0\}$, a collection of copies of \hat{D}_U , such that $x_j \in W_j^0$ for each $j = 1, \dots, k_0$.

Set $t := 0$ and repeat the following steps until STOP:

1. If there is a droplet $W_i^t \in \mathcal{W}^t$ and an $x \in \mathbb{Z}^2$ such that the set

$$W_i^t \cup (x + \hat{D}_U) \cup \mathbb{H}_u$$

is strongly connected, then set

$$\mathcal{W}^{t+1} := (\mathcal{W}^t \setminus \{W_i^t\}) \cup \{J_u(W_i^t)\},$$

and set $t := t + 1$.

2. If not, but there are two sets $W_i^t, W_j^t \in \mathcal{W}^t$ and an $x \in \mathbb{Z}^2$ such that the set

$$W_i^t \cup W_j^t \cup (x + \hat{D}_U) \cup \mathbb{H}_u$$

is strongly connected, then set

$$\mathcal{W}^{t+1} := (\mathcal{W}^t \setminus \{W_i^t, W_j^t\}) \cup \{J_u(W_i^t \cup W_j^t)\},$$

and set $t := t + 1$.

3. If not, but there are two droplets $W_i^t, W_j^t \in \mathcal{W}^t$ and an $x \in \mathbb{Z}^2$ such that the set

$$W_i^t \cup W_j^t \cup (x + \hat{D}_U)$$

is strongly connected, then set

$$\mathcal{W}^{t+1} := (\mathcal{W}^t \setminus \{W_i^t, W_j^t\}) \cup \{D(W_i^t \cup W_j^t)\},$$

and set $t := t + 1$.

4. Otherwise set $T := t$ and STOP.

The output of the algorithm is the family $\mathcal{W} := \{W_1^T, \dots, W_k^T\}$.

Thus, at each step of the algorithm we have a collection \mathcal{W}^t of \mathcal{S}_U -droplets and u -icebergs; we either take a droplet near \mathbb{H}_u and replace it by the smallest u -iceberg containing it, or we take two nearby sets in our collection, and replace them by either the smallest u -iceberg containing their union (if they are sufficiently close to \mathbb{H}_u), or by the smallest droplet containing their union, otherwise.

Definition 6.16. We say that $\mathcal{W} = \{W_1, \dots, W_k\}$ is a u -iceberg-cover of a finite set K if \mathcal{W} is a possible output of the u -iceberg algorithm with input K . We say that an iceberg J is u -iceberg-covered if $\mathcal{W} = \{J\}$ is a u -iceberg-cover of $J \cap A$.

We can now prove our extremal result for icebergs; the lemma is illustrated in Figure 4. Let $\sigma(u)$ denote the angle (in radians) between u and u^* .

Lemma 6.17 (Extremal lemma for u -iceberg covers). *Let $u \in [u_0, u^*]$, let J be a u -iceberg covered u -iceberg, and let $\gamma = |J \cap A|$. Then*

$$w(J) \leq O(\gamma/\sigma(u)) \quad \text{and} \quad h(J) \leq O(\gamma),$$

where the implicit constants depend on \mathcal{U} , but not on J , γ or u .

Proof. Note first that

$$T = O(\gamma), \tag{18}$$

since at all but at most γ steps of the algorithm, $|\mathcal{W}^t|$ is reduced by 1.

Let \mathcal{D}^t and \mathcal{J}^t denote, respectively, the collections of droplets and u -icebergs in \mathcal{W}^t , so $\mathcal{W}^t = \mathcal{D}^t \cup \mathcal{J}^t$. In order to prove the bound on the width of J in the lemma, we claim that, for each $t \leq T$,

$$\sum_{D^t \in \mathcal{D}^t} h(D^t) + \sigma(u) \sum_{J^t \in \mathcal{J}^t} w(J^t) = O(t + \gamma) = O(\gamma). \quad (19)$$

The second equality is just (18). To see the first, note that the claim is clearly true when $t = 0$, and that at each step the left-hand side of (19) increases by at most $O(1)$. When a droplet and a u -iceberg, or two u -icebergs, are replaced by a u -iceberg (as in step 2 of the u -iceberg algorithm), or when two droplets are replaced by another droplet (as in step 3 of the algorithm), this is clear, because the sums individually increase by at most $O(1)$ (as in (14)). When a droplet D^t is replaced by a u -iceberg (as in step 1 of the algorithm), the first sum in (19) decreases by $h(D^t)$, and the second increases by at most $\sigma(u) \cdot (h(D^t)/\sigma(u) + O(1))$. This proves (19), and hence, since the first sum is non-negative and the output of the u -iceberg algorithm is the single iceberg J , that

$$\sigma(u) \cdot w(J) = O(\gamma),$$

which is the first part of the lemma.

For the second part, we claim that, for each $t \leq T$,

$$\sum_{D^t \in \mathcal{D}^t} (\sigma(u) \cdot w(D^t) + h(D^t)) + \sum_{J^t \in \mathcal{J}^t} h(J^t) = O(t + \gamma) = O(\gamma). \quad (20)$$

As in the previous case, the second equality is just (18), and the claim is trivial when $t = 0$. Therefore it is enough to prove that the left-hand side of (20) increases by at most $O(1)$ at each step. When step 2 or 3 of the algorithm is applied, this is clear as before. When step 1 of the algorithm is applied, and a droplet D^t is replaced by a u -iceberg, the first sum decreases by $\sigma(u) \cdot w(D^t) + h(D^t)$ and the second sum increases by $\sigma(u) \cdot w(D^t) + h(D^t) + O(1)$. Thus we have

$$h(J) = O(\gamma),$$

and this completes the proof of the lemma. \square

7. THE LOWER BOUND FOR BALANCED FAMILIES

Using the tools developed in the previous section, we can now prove the lower bound in Theorem 1.4 when \mathcal{U} is a balanced update family.

Theorem 7.1. *Let \mathcal{U} be a balanced critical update family. Then*

$$p^\alpha \log \tau = \Omega(1)$$

with high probability as $p \rightarrow 0$.

We shall prove Theorem 7.1 by showing that if the origin is infected by time t then one of three events must have occurred: either there exists an α -covered droplet of diameter roughly $\log t$ within distance t of the origin, or there exists a smaller α -covered droplet very close to the origin, or the origin is infected despite lying outside the α -cover. We bound the probability of the first and second events using Lemma 6.4, and the probability of the third event using Lemma 6.5.

In this section we abuse notation slightly by writing $D(k)$ for the unique minimal \mathcal{S}_B -droplet all of whose sides are at ℓ_2 distance at least λk from the origin, where $k \in \mathbb{N}$. (This is an abuse of notation because we have previously defined $D(K)$ to be the minimal droplet containing a finite set K . In this section we shall only use the new notation.)

Lemma 7.2. *Let $\varepsilon > 0$ be sufficiently small, and let $t \in \mathbb{N}$ and $p \in (0, 1)$ be such that*

$$p \leq \left(\frac{\varepsilon}{\log t} \right)^{1/\alpha}.$$

Then, with high probability as $p \rightarrow 0$, there does not exist an α -covered droplet $D \subset D(t)$ such that $\log t \leq \text{diam}(D) \leq 3 \log t$.

Proof. By Lemma 6.4, there exists $\delta > 0$ such that, if D is an α -covered droplet with diameter at least $\log t$, then $D \cap A$ contains at least $\delta \log t$ disjoint α -clusters. Noting that D contains $O((\text{diam } D)^2)$ distinct α -clusters, it follows that if D is a droplet with diameter in the range $\log t \leq \text{diam}(D) \leq 3 \log t$ then the probability D is α -covered is at most

$$\binom{O(\log t)^2}{\delta \log t} p^{\alpha \delta \log t} \leq \left(O(p^\alpha \log t) \right)^{\delta \log t} \leq \frac{1}{t^3},$$

since ε is sufficiently small. Since there are $t^2(\log t)^{O(1)}$ droplets in $D(t)$ having diameter at most $3 \log t$, the probability that there exists an α -covered droplet $D \subset D(t)$ such that $\log t \leq \text{diam}(D) \leq 3 \log t$ is at most

$$t^2 \cdot (\log t)^{O(1)} \cdot t^{-3} = o(1),$$

as required. \square

Lemma 7.3. *Let $\varepsilon > 0$ be sufficiently small, and let $t \in \mathbb{N}$ and $p \in (0, 1)$ be such that*

$$p \leq \left(\frac{\varepsilon}{\log t} \right)^{1/\alpha}.$$

Then, with high probability as $p \rightarrow 0$, there does not exist an α -covered droplet $D \subset D(\log t)$ containing the origin.

Proof. We begin as in the previous lemma: by Lemma 6.4, there exists $\delta > 0$ such that, if D is an α -covered droplet with diameter at least k , then $D \cap A$ contains at least δk disjoint α -clusters. Since a droplet with diameter in the interval $[k, k + 1]$

contains $O(k^2)$ distinct α -clusters, it follows that such a droplet D is α -covered with probability at most

$$\binom{O(k^2)}{\delta k} p^{\alpha \delta k} \leq (O(kp^\alpha))^{\delta k} = \left(\frac{O(\varepsilon k)}{\log t} \right)^{\delta k}.$$

There are $k^{O(1)}$ droplets with diameter in the interval $[k, k+1]$ containing the origin. Hence, the probability that the origin is contained in an α -covered droplet $D \subset D(\log t)$ is at most

$$\sum_{k=1}^{O(\log t)} k^{O(1)} \left(\frac{O(\varepsilon k)}{\log t} \right)^{\delta k}.$$

By splitting up the sum into the first $(\log \log t)^2$ terms and then the remaining terms, this probability is at most

$$(\log \log t)^{O(1)} \left(\frac{(\log \log t)^{O(1)}}{\log t} \right)^\delta + (\log t)^{O(1)} O(\varepsilon)^{\delta (\log \log t)^2} = o(1), \quad (21)$$

as required. \square

Lemma 7.4. *Let $\varepsilon > 0$ be sufficiently small, and let $t \in \mathbb{N}$ and $p \in (0, 1)$ be such that*

$$p \leq \left(\frac{\varepsilon}{\log t} \right)^{1/\alpha}.$$

Then, with high probability as $p \rightarrow 0$, either there exists an α -covered droplet $D \subset D(t)$ containing the origin, or $\mathbf{0} \notin [D(t) \cap A]$.

Proof. Let $\mathcal{D} = \{D_1, \dots, D_k\}$ be an α -cover of $D(t) \cap A$. If

$$\mathbf{0} \in [D(t) \cap A] \setminus (D_1 \cup \dots \cup D_k),$$

then, by Lemma 6.5, there must be an element of A within distance ρ of the origin. But this has probability $O(p) = o(1)$, as required. \square

Proof of Theorem 7.1. Let $\varepsilon > 0$ be a sufficiently small constant, and set

$$p = \left(\frac{\varepsilon}{\log t} \right)^{1/\alpha}.$$

We shall show that $\tau > t$ with high probability as $p \rightarrow 0$.

Observe first that if $\tau \leq t$ then $\mathbf{0} \in [D(t) \cap A]$, since otherwise there would have to be a path of successive infections from $D(t)^c$ to the origin, and any such path has length greater than t . Thus, by Lemma 7.4, the probability that $\tau \leq t$ and there does not exist an α -covered droplet $D \subset D(t)$ containing the origin is $o(1)$.

Now, by Lemma 7.3, the probability that such a droplet exists with $\text{diam}(D) \leq \log t$ is $o(1)$. On the other hand, if there exists an α -covered droplet D with $\text{diam}(D) \geq \log t$ then, by Lemma 6.3, there exists an α -covered droplet $D' \subset D$ such that $\log t \leq \text{diam}(D') \leq 3 \log t$. But, by Lemma 7.2, the probability that such a droplet exists is $o(1)$. Hence $\tau > t$ with high probability, as required. \square

8. THE LOWER BOUND FOR UNBALANCED FAMILIES

In this section we shall prove the following theorem, and hence complete the proof of Theorem 1.4.

Theorem 8.1. *Let \mathcal{U} be an unbalanced critical update family. Then*

$$p^\alpha \left(\log \frac{1}{p} \right)^{-2} \log \tau = \Omega(1)$$

with high probability as $p \rightarrow 0$.

Throughout the section we assume that \mathcal{U} is unbalanced, and that droplets are taken with respect to the set $\mathcal{S}_U = \{u^*, -u^*, u^r, u^l\}$, where

$$\min \{ \alpha(u^*), \alpha(-u^*) \} \geq \alpha + 1 \quad \text{and} \quad \min \{ \bar{\alpha}(u^l), \bar{\alpha}(u^r) \} = \alpha,$$

and u^l and u^r are contained in opposite semicircles separated by u^* and $-u^*$, with u^l to the left and u^r to the right of u^* . We also let $\xi > 0$ be a sufficiently small constant (which will depend on the constant $\delta(2\alpha + 1)$ defined below; see the remark after Definition 8.3), and we fix

$$\eta := \frac{1}{10\alpha}.$$

The main step in the proof of Theorem 8.1 is an upper bound on the probability that a critical droplet is internally spanned. Recall from Definition 2.5 that in this section a droplet D is said to be *critical* if its dimensions satisfy either

$$(T) \quad w(D) \leq p^{-\alpha-1/5} \text{ and } \frac{\xi}{p^\alpha} \log \frac{1}{p} \leq h(D) \leq \frac{3\xi}{p^\alpha} \log \frac{1}{p}, \text{ or}$$

$$(L) \quad p^{-\alpha-1/5} \leq w(D) \leq 3p^{-\alpha-1/5} \text{ and } h(D) \leq \frac{\xi}{p^\alpha} \log \frac{1}{p}.$$

The precise bound that we shall prove is the following. (Recall again that $I^\times(D)$ is the event that the \mathcal{S}_U -droplet D is internally spanned.)

Lemma 8.2. *There exists $\delta > 0$ such that if D is a critical droplet then*

$$\mathbb{P}_p(I^\times(D)) \leq \exp \left(-\frac{\delta}{p^\alpha} \left(\log \frac{1}{p} \right)^2 \right).$$

We build up the proof of Lemma 8.2 gradually via an induction argument, at each step of which we bound the probability that droplets of certain (increasingly large) sizes are internally spanned.

Definition 8.3. For $\beta_1, \beta_2 \in \mathbb{N}$, let $\text{IH}(\beta_1, \beta_2)$ be the following statement:

There exists $\delta = \delta(\beta_1 + \beta_2) > 0$ such that the following holds. Let D be a droplet such that

$$w(D) \leq p^{-\beta_1(1-2\eta)-\eta} \quad \text{and} \quad h(D) \leq p^{-\beta_2(1-2\eta)-\eta}.$$

Then

$$\mathbb{P}_p(I^\times(D)) \leq p^{\delta \max\{w(D), h(D)\}}. \tag{22}$$

The constants $\delta(\beta)$, for $2 \leq \beta \leq 2\alpha + 1$, and the constants δ (from Lemma 8.2), ξ (from the definition of a critical droplet), and ε (which will be used in the proof of Theorem 8.1 in Section 8.6), will be chosen so that

$$1 \gg \delta(2) \gg \cdots \gg \delta(2\alpha + 1) \gg \xi \gg \delta \gg \varepsilon > 0. \quad (23)$$

During the course of the inductive proof of Lemma 8.4, we shall introduce two further sequences of constants. The relationships between these new constants and those in (23) will be set out explicitly in (30) and (31).

We mention briefly that we would prefer the width and height conditions in Definition 8.3 to be $w(D) \leq p^{-\beta_1(1-\eta)}$ and $h(D) \leq p^{-\beta_2(1-\eta)}$ respectively, but for technical reasons we cannot quite square the bound on the width between $\beta_1 = 1$ and $\beta_1 = 2$; this is why the conditions take the less elegant form stated.

The specific induction statements that we shall prove are:

$$\begin{aligned} \text{IH}(\beta, \beta) &\Rightarrow \text{IH}(\beta + 1, \beta) && \text{for all } 1 \leq \beta \leq \alpha; \\ \text{IH}(\beta, \beta) &\Rightarrow \text{IH}(\beta, \beta + 1) && \text{for all } 1 \leq \beta \leq \alpha; \\ (\text{IH}(\beta + 1, \beta) \wedge \text{IH}(\beta, \beta + 1)) &\Rightarrow \text{IH}(\beta + 1, \beta + 1) && \text{for all } 1 \leq \beta \leq \alpha - 1. \end{aligned}$$

Note that $\text{IH}(1, 1)$ is an immediate consequence of Lemma 6.11, and therefore together these statements will be enough to prove the following lemma.

Lemma 8.4. *The assertions $\text{IH}(\alpha + 1, \alpha)$ and $\text{IH}(\alpha, \alpha + 1)$ both hold.*

While Lemma 8.4 alone does not imply the bound on internally spanned critical droplets in Lemma 8.2, the techniques and lemmas that we use to prove the former are the same as those we use to deduce from it the latter.

The steps in the induction are of two types: *horizontal steps* of the form

$$' \text{IH}(\beta_1, \beta_2) \Rightarrow \text{IH}(\beta_1 + 1, \beta_2) ',$$

and *vertical steps* of the form

$$' \text{IH}(\beta_1, \beta_2) \Rightarrow \text{IH}(\beta_1, \beta_2 + 1) '.$$

Common to both is the key idea of *crossings*. Roughly speaking, these are events that say that it is possible to ‘cross’ a parallelogram of sites from one side to the other with ‘help’ from one of the sides in the form of an infected half-plane. The events should be thought of in the context of a growing droplet: a combination of crossings events, one for each side of the droplet, enable an internally filled droplet to grow into a larger internally spanned droplet. We obtain bounds for the probabilities of crossings by showing that, to a certain level of precision, the most likely way these events could occur is via the droplet (or half-plane) advancing row-by-row, rather than via the merging of many smaller droplets. One could think of this as saying that the growth mechanism we used to prove the upper bound for unbalanced families in Theorem 5.1, which was indeed row-by-row, was essentially the ‘correct’ mechanism. For vertical crossings in the case of models with drift, our proof will

make use of the results of Section 6.3 on the iceberg algorithm to bound the range of the \mathcal{U} -bootstrap process in directions close to $\pm u^*$. Full statements and proofs of the crossing lemmas, together with precise definitions, are given in Section 8.3.

For the horizontal steps, in addition, we require the use of ‘hierarchies’ to bound the extent of sideways growth at any given step. These are by now a standard tool in the bootstrap percolation literature, so we omit many of the details.

There are six subsections in this section, which deal with the following aspects of the proof: in the first we establish the hierarchies framework; in the second we derive a bound on the range of the \mathcal{U} -bootstrap process in the geometry of the u -norm; in the third we prove the crossing lemmas; in the fourth we assemble the different parts of the induction statement and prove Lemma 8.4; in the fifth we deduce Lemma 8.2, which is the bound for internally spanned critical droplets; and in the sixth and final subsection we complete the proof of Theorem 8.1.

8.1. Hierarchies. The use of hierarchies to control the formation of critical droplets was introduced in [27] and has since developed into a standard technique in bootstrap percolation [3, 4, 19, 20, 25]. In this subsection we recall many of the standard definitions and lemmas, making only minor adaptations along the way to suit the general model. We are relatively brief with the details, referring the reader instead to the paper of Holroyd [27] and the recent refinements in [20, 25] for a more extensive introduction to the method.

The key result of this subsection is Lemma 8.9, which gives an upper bound for the probability that a droplet D is internally spanned in terms of the family of hierarchies of D .

Given a directed graph G and a vertex $v \in V(G)$, we write $N_G^\rightarrow(v)$ for the set of out-neighbours of v in G .

Definition 8.5. Let D be an \mathcal{S}_U -droplet. A *hierarchy* \mathcal{H} for D is an ordered pair $\mathcal{H} = (G_{\mathcal{H}}, D_{\mathcal{H}})$, where $G_{\mathcal{H}}$ is a directed rooted tree such that all of its edges are directed away from the root v_{root} , and $D_{\mathcal{H}}: V(G_{\mathcal{H}}) \rightarrow 2^{\mathbb{Z}^2}$ is a function that assigns to each vertex of $G_{\mathcal{H}}$ an \mathcal{S}_U -droplet, such that the following conditions are satisfied:

- (i) the root vertex corresponds to D , so $D_{\mathcal{H}}(v_{\text{root}}) = D$;
- (ii) each vertex has out-degree at most 2;
- (iii) if $v \in N_{G_{\mathcal{H}}}^\rightarrow(u)$ then $D_{\mathcal{H}}(v) \subset D_{\mathcal{H}}(u)$;
- (iv) if $N_{G_{\mathcal{H}}}^\rightarrow(u) = \{v, w\}$ then $D_{\mathcal{H}}(u) \in \langle D_{\mathcal{H}}(v) \cup D_{\mathcal{H}}(w) \rangle$.

Condition (iv) is equivalent to the statement that $D_{\mathcal{H}}(v) \cup D_{\mathcal{H}}(w)$ is strongly connected and that $D_{\mathcal{H}}(u)$ is the smallest droplet containing their union. We shall usually abbreviate $D_{\mathcal{H}}(u)$ to D_u .

The next definition controls the absolute and relative sizes of the droplets corresponding to vertices of $G_{\mathcal{H}}$, which in turn allows us to control the number of hierarchies. In order to limit the number of hierarchies as much as possible, we

choose the step size to be as large as possible, subject to the condition that we can control the probability of each step.

Definition 8.6. Fix $\beta \in \mathbb{N}$. A hierarchy \mathcal{H} for a droplet D is *good* if it satisfies the following conditions for each $u \in V(G_{\mathcal{H}})$:

- (v) u is a leaf if and only if $w(D_u) \leq p^{-\beta(1-2\eta)-\eta}$;
- (vi) if $N_{G_{\mathcal{H}}}^{\rightarrow}(u) = \{v\}$ and $|N_{G_{\mathcal{H}}}^{\rightarrow}(v)| = 1$ then
$$p^{-\beta(1-2\eta)-\eta}/2 \leq w(D_u) - w(D_v) \leq p^{-\beta(1-2\eta)-\eta};$$
- (vii) if $N_{G_{\mathcal{H}}}^{\rightarrow}(u) = \{v\}$ and $|N_{G_{\mathcal{H}}}^{\rightarrow}(v)| \neq 1$ then $w(D_u) - w(D_v) \leq p^{-\beta(1-2\eta)-\eta}$;
- (viii) if $N_{G_{\mathcal{H}}}^{\rightarrow}(u) = \{v, w\}$ then $w(D_u) - w(D_v) \geq p^{-\beta(1-2\eta)-\eta}/2$.

Next we relate the abstract family of good hierarchies defined above to the initial set A of infected sites and to the \mathcal{U} -bootstrap process. Given nested \mathcal{S}_U -droplets $D \subset D'$, we write $\Delta(D, D')$ for the event that D' is internally spanned given that D is internally filled. That is,

$$\Delta(D, D') := \{D' \in \langle D \cup (D' \cap A) \rangle\}.$$

The final two conditions below ensure that a good hierarchy for an internally spanned droplet D accurately represents the growth of the initial sites $D \cap A$.

Definition 8.7. A hierarchy \mathcal{H} for D is *satisfied* by A if the following events all occur *disjointly*:

- (ix) if v is a leaf then D_v is internally spanned by A ;
- (x) if $N_{G_{\mathcal{H}}}^{\rightarrow}(u) = \{v\}$ then $\Delta(D_v, D_u)$ occurs.

Having established all of the properties of hierarchies that we need, we now show that there exists a good and satisfied hierarchy for every internally spanned droplet. The proof is almost identical to Propositions 31 and 33 of [27], which deal with the 2-neighbour setting, except that here we use the spanning algorithm in place of the rectangles process.

Lemma 8.8. *Let D be an \mathcal{S}_U -droplet internally spanned by A . Then there exists a good and satisfied hierarchy for D .*

Proof. In order to prove the lemma we consider a suitable ‘contraction’ of the tree given by the spanning algorithm. To that end, let $\mathcal{D} = \langle D \cap A \rangle$, and note that $D \in \mathcal{D}$ since D is internally spanned. The proof will be by induction on $w(D)$, so note first that if $w(D) \leq p^{-\beta(1-2\eta)-\eta}$ then we may take $V(G_{\mathcal{H}}) = \{v_{\text{root}}\}$.

For the induction step, first we claim that there exists a pair of sequences,

$$D \cap A \supset K_0 \supset K_1 \supset \cdots \supset K_m \quad \text{and} \quad D = D_0 \supset D_1 \supset \cdots \supset D_m,$$

such that $|K_m| = 1$ and such that for every $1 \leq i \leq m$,

$$D_i = D([K_i]) \quad \text{and} \quad [K_i] \cup [K_{i-1} \setminus K_i] \text{ is strongly connected.}$$

To construct these sequences, we run the spanning algorithm backwards, choosing at each step the larger of the two droplets. Indeed, since $D \in \langle D \cap A \rangle$, there exists a set $K_0 \subset D \cap A$ such that $[K_0]$ is strongly connected and $D = D([K_0])$. Now, given K_{i-1} such that $[K_{i-1}]$ is strongly connected, the spanning algorithm gives a (non-trivial) partition $K_i \cup K'_i$ of K_{i-1} such that $[K_i]$, $[K'_i]$ and $[K_i] \cup [K'_i]$ are all strongly connected, by Lemma 6.12. Set $D_i = D([K_i])$ and $D'_i = D([K'_i])$, where $w(D_i) \geq w(D'_i)$.

Now, let $s \geq 1$ be minimal such that either

$$w(D_s) \leq p^{-\beta(1-2\eta)-\eta} \quad \text{or} \quad w(D) - w(D_s) \geq \frac{p^{-\beta(1-2\eta)-\eta}}{2},$$

and attach a vertex u corresponding to D_s to the root. If $w(D_s) \leq p^{-\beta(1-2\eta)-\eta}$ and $w(D) - w(D_s) \leq p^{-\beta(1-2\eta)-\eta}$, then our construction of \mathcal{H} is complete. If $p^{-\beta(1-2\eta)-\eta}/2 \leq w(D) - w(D_s) \leq p^{-\beta(1-2\eta)-\eta}$, then we use the induction hypothesis to construct a good and satisfied (by K_s) hierarchy \mathcal{H}' for D_s , and identify u with the root of \mathcal{H}' . Finally, if $w(D) - w(D_s) \geq p^{-\beta(1-2\eta)-\eta}$ then, by the minimality of s , we have

$$w(D_{s-1}) - w(D'_s) \geq w(D_{s-1}) - w(D_s) \geq \frac{p^{-\beta(1-2\eta)-\eta}}{2}.$$

In this case we add a vertex v between u and the root, corresponding to D_{s-1} , and add another vertex w attached to v , corresponding to D'_s . Now, using the induction hypothesis, we construct good and satisfied (by K_s and $K_{s-1} \setminus K_s$ respectively) hierarchies \mathcal{H}' and \mathcal{H}'' for D_s and D'_s , and identify u and w with the roots of \mathcal{H}' and \mathcal{H}'' . It is straightforward to check that the hierarchies thus constructed satisfy conditions (i)–(x), as required. \square

We emphasize that the existence of a good and satisfied hierarchy for D does *not* imply that D is internally spanned, since the intersection of the events $I^\times(D_v)$ and $\Delta(D_v, D_u)$ does not imply that D_u is internally spanned, and since we do not insist that $[(D_v \cup D_w) \cap A]$ is strongly connected whenever $N_{G_{\mathcal{H}}}^\rightarrow(u) = \{v, w\}$. It is one of the key ideas of the proof that these approximations do not affect the probability estimates too much.

The first step in the proof of Lemma 8.2 for type (L) critical droplets is the following fundamental bound on the probability that a droplet is internally spanned.

Let us write \mathcal{H}_D for the set of all good hierarchies for D , and $L(\mathcal{H})$ for the set of leaves of $G_{\mathcal{H}}$. We write $\prod_{u \rightarrow v}$ for the product over all pairs $\{u, v\} \subset V(G_{\mathcal{H}})$ such that $N_{G_{\mathcal{H}}}^\rightarrow(u) = \{v\}$.

Lemma 8.9. *Let D be a droplet. Then*

$$\mathbb{P}_p(I^\times(D)) \leq \sum_{\mathcal{H} \in \mathcal{H}_D} \left(\prod_{u \in L(\mathcal{H})} \mathbb{P}_p(I^\times(D_u)) \right) \left(\prod_{u \rightarrow v} \mathbb{P}_p(\Delta(D_v, D_u)) \right). \quad (24)$$

Proof of Lemma 8.9. Since the events $I^\times(D_u)$ for $u \in L(\mathcal{H})$ and $\Delta(D_v, D_u)$ for $u \rightarrow v$ are increasing and occur disjointly, this is immediate from Lemma 8.8 and the van den Berg-Kesten inequality (Lemma 2.11). \square

In order to use Lemma 8.9 we must bound three quantities:

- the probability of the event $I^\times(D_u)$ for each $u \in L(\mathcal{H})$;
- the probability of the event $\Delta(D_v, D_u)$ for every pair $u, v \in V(G_{\mathcal{H}})$ with $N_{G_{\mathcal{H}}}^{\rightarrow}(u) = \{v\}$;
- the number of good hierarchies for D .

Our bound on the first probability follows immediately from the induction hypothesis. We shall bound the second in Section 8.3, again using the induction hypothesis, but this time the proof is considerably more difficult. To count the good hierarchies, we partition the set \mathcal{H}_D according to the number of ‘big seeds’, as follows:

Definition 8.10. If \mathcal{H} is a hierarchy and $v \in L(\mathcal{H})$, then we say that D_v is a *seed* of \mathcal{H} . If moreover $w(D_v) \geq p^{-\beta(1-2\eta)-\eta}/3$, then we say that D_v is a *big seed* of \mathcal{H} .

The concept of a big seed was first used in [25]. We denote by $b(\mathcal{H})$ the number of big seeds in a hierarchy \mathcal{H} , and by \mathcal{H}_D^b the set of all good hierarchies for D that have exactly b big seeds. Finally, let $d(\mathcal{H})$ denote the depth of the tree $G_{\mathcal{H}}$. Thus, $d(\mathcal{H})$ is the maximum length of a path from the root to a leaf in $G_{\mathcal{H}}$.

Lemma 8.11. *Let D be an \mathcal{S}_U -droplet with $w(D) \leq p^{-(\beta+1)(1-2\eta)-\eta}$ and $h(D) = p^{-O(1)}$. Then*

$$|\mathcal{H}_D^b| \leq \exp \left[O \left(b \cdot w(D) \cdot p^{\beta(1-2\eta)+\eta} \log \frac{1}{p} \right) \right].$$

Proof. By the definition of a good hierarchy, every vertex $v \in G_{\mathcal{H}}$ that is not a leaf must lie above a big seed. This simple but key observation, which was made in [25], immediately implies that

$$|V(G_{\mathcal{H}})| \leq 2 \cdot b(\mathcal{H}) \cdot d(\mathcal{H}). \quad (25)$$

We claim that

$$d(\mathcal{H}) = O(w(D) \cdot p^{\beta(1-2\eta)+\eta}). \quad (26)$$

Indeed, this follows from the fact that every two steps up $G_{\mathcal{H}}$, the width of the corresponding rectangle increases by $\Omega(p^{-\beta(1-2\eta)-\eta})$. We therefore have $|V(G_{\mathcal{H}})| = O(b \cdot w(D) \cdot p^{\beta(1-2\eta)+\eta})$ for every $\mathcal{H} \in \mathcal{H}_D^b$.

Now, the number of choices for the tree $G_{\mathcal{H}}$ is at most $2^{O(N)}$, where N is our bound on $|V(G_{\mathcal{H}})|$. Moreover, for each $u \in V(G_{\mathcal{H}})$, there are at most $p^{-O(1)}$ possible droplets D_u . Hence

$$|\mathcal{H}_D^b| \leq \exp \left[O \left(b \cdot w(D) \cdot p^{\beta(1-2\eta)+\eta} \log \frac{1}{p} \right) \right],$$

as required. \square

8.2. The range of unbalanced models with drift. In this short section we assume that \mathcal{U} is an unbalanced model with drift and we use the results about icebergs from Section 6.3 to prove a bound (see Lemma 8.12) on the range of the \mathcal{U} -bootstrap process helped by a half-plane \mathbb{H}_u .

Recall from Section 6.3 that if $\alpha^-(u^*) = \infty$ then we choose $u_0 \in S^1$ to the left of and sufficiently close to u^* , so in particular $\alpha^-(u) = \infty$ for every $u \in [u_0, u^*]$. Similarly, if $\alpha^+(-u^*) = \infty$ then we choose a corresponding $u'_0 \in S^1$ to the right of and sufficiently close to $-u^*$. Set

$$\mathcal{S}_U^+ := \begin{cases} [u_0, u^*] & \text{if } \alpha^-(u^*) = \infty \\ \{u^*\} & \text{otherwise} \end{cases} \quad \text{and} \quad \mathcal{S}_U^- := \begin{cases} [-u^*, u'_0] & \text{if } \alpha^+(-u^*) = \infty \\ \{-u^*\} & \text{otherwise,} \end{cases}$$

and set

$$\mathcal{S}_U^\pm := \mathcal{S}_U^+ \cup \mathcal{S}_U^- \quad \text{and} \quad \mathcal{S}'_U := \{u^l, u^r\} \cup \mathcal{S}_U^\pm.$$

Recall also that for each $u \in \mathcal{S}_U^+$, we defined $\sigma(u)$ to be the angle between u and u^* , and similarly for each $u \in \mathcal{S}_U^-$.

Now let $u \in \mathcal{S}'_U$ and define a norm $\|\cdot\|_u$ on \mathbb{R}^2 as follows:

$$\|x\|_u := \begin{cases} |\langle x, u^* \rangle| + \sigma(u) |\langle x, u^\perp \rangle| & \text{if } u \in \mathcal{S}_U^\pm \setminus \{u^*, -u^*\}, \\ \|x\|_2 & \text{otherwise.} \end{cases} \quad (27)$$

We record for later use the easy inequalities:

$$|\langle x, u \rangle| = \cos \sigma \cdot |\langle x, u^* \rangle| + \sin \sigma \cdot |\langle x, u^\perp \rangle| \leq \|x\|_u \leq 2 \cdot \|x\|_2 \quad (28)$$

for every $x \in \mathbb{R}^2$. Let $\rho: \mathcal{S}'_U \times \mathbb{N} \rightarrow \mathbb{R}$ be the function given by

$$\rho(u, \gamma) := \max \left\{ \|y - Y\|_u : |Y| = \gamma - 1, y \in [\mathbb{H}_u \cup Y] \setminus \mathbb{H}_u \right\}, \quad (29)$$

if the maximum exists, and ∞ otherwise. Note that, by the definition of $\bar{\alpha}(u)$, we have $\rho(u, \gamma) = \infty$ if and only if $\gamma > \bar{\alpha}(u)$.

The key property that we need for the vertical crossings lemma, and the main result of this section, is the following bound on $\rho(u, \gamma)$, which is uniform in u .

Lemma 8.12. *Let $u \in \mathcal{S}'_U$ and $\gamma \in \mathbb{N}$, with $\gamma \leq \bar{\alpha}(u)$. Then $\rho(u, \gamma)$ is bounded above by a function of γ and \mathcal{U} only.*

Proof. Note first that since \mathcal{S}_U is finite (in fact, it has size 4), it is sufficient to prove the lemma assuming $u \in \mathcal{S}_U^\pm \setminus \{u^*, -u^*\}$, and hence, by symmetry, only for $u \in \mathcal{S}_U^+ \setminus \{u^*\}$.

So suppose that $u \in \mathcal{S}_U^+ \setminus \{u^*\}$ and note that this immediately implies $\alpha^-(u^*) = \infty$. Let $K \subset \mathbb{Z}^2$ be a set of size $\gamma - 1$, and let \mathcal{W} be a u -iceberg cover of K . The set

$$\mathbb{H}_u \cup \bigcup_{W \in \mathcal{W}} W$$

contains K and is closed, by Lemma 6.14, and because the algorithm has terminated with the collection \mathcal{W} . Furthermore, if $y \in W$ for some $W \in \mathcal{W}$, then $\|x - y\|_u = O(\gamma)$ for some $x \in W \cap K$, by Lemma 6.17. \square

8.3. Crossing lemmas. This subsection is the most important of the paper. Our primary aim is to bound the probabilities of certain ‘crossing’ events, with a view to two specific applications. The horizontal crossings lemma (Lemma 8.15) will enable us to bound the probability of events of the form $\Delta(D, D')$, which in turn allows us to bound the probability that ‘long’ droplets are internally spanned using the hierarchies bound of Lemma 8.9. The vertical crossings lemma (Lemma 8.16) will enable us to bound (directly) the probability that ‘tall’ droplets are internally spanned.

Since there is significant overlap between the proofs for ‘horizontal’ and ‘vertical’ crossings, it will be convenient to work in the following (slightly) more general framework.

Definition 8.13. Let $u \in \mathcal{S}'_U$. A finite set is a u -strip if it is a \mathcal{T} -droplet, where $\mathcal{T} = \{u, -u, v, -v\}$ and either

- $u \in \{u^l, u^r\}$ and $v = u^*$ (a *horizontal strip*), or
- $u \in \mathcal{S}_U^\pm = \mathcal{S}'_U \setminus \{u^l, u^r\}$ and $v = u^\perp$ (a *vertical strip*).

Although it is convenient to define u -strips in terms of \mathcal{T} -droplets, we stress again that *all* sets described in this section as ‘droplets’ without reference to a set \mathcal{T} are assumed to be \mathcal{S}_U -droplets.

Recall that if D is a \mathcal{T} -droplet and $u \in \mathcal{T}$, then we define $\partial(D, u)$, the u -side of D , to be the set $D \cap \ell_u(i)$, where i is maximal so that this set is non-empty.

Definition 8.14. Let $u \in S^1$, let S be a u -strip, and let $x \in \partial(S, -u)$. We say that S is u -crossed if there exists a strongly connected set in $[\mathbb{H}_u(x) \cup (S \cap A)]$ that intersects both $\mathbb{H}_u(x)$ and $\partial(S, u)$.

Unless the precise position of the u -strip is important, we shall assume without further comment that the $(-u)$ -side of the u -strip is a subset of ℓ_u , so that we may take the x in the previous definition to be the origin.

Before continuing with the results of this subsection, we give a more complete account of the relationships between the different constants of this section (leading up to the proof of Theorem 8.1) than that given in (23). We mentioned that during the course of the inductive proof of Lemma 8.4 two further sequences of constants would be defined, in addition to the $\delta(\beta)$ already introduced in Definition 8.3. These sequences are $\delta'(2), \dots, \delta'(2\alpha + 1)$, which appear in the statements of Lemmas 8.15 and 8.16, and $\kappa_0(2), \dots, \kappa_0(2\alpha + 1)$, which appear in Definition 8.17. These constants will be chosen to have the following relative sizes. First, for each $2 \leq \beta \leq 2\alpha$,

$$1 \gg \delta(\beta) \gg \frac{1}{\kappa_0(\beta)} \gg \delta'(\beta) \gg \delta(\beta + 1) > 0, \quad (30)$$

and second,

$$1 \gg \delta(2\alpha + 1) \gg \frac{1}{\kappa_0(2\alpha + 1)} \gg \delta'(2\alpha + 1) \gg \xi \gg \delta \gg \varepsilon > 0. \quad (31)$$

Note that these two sets of relations subsume those in (23).

The main results of this subsection are the following two lemmas. One may think of the lemmas as exchanging bounds on the probability that a droplet is internally spanned for bounds on the probability that similarly sized u -strips are u -crossed, for some u . (It may be helpful, therefore, to think of the $\delta'(\beta)$ as being to crossing u -strips as the $\delta(\beta)$ are to internally spanning \mathcal{S}_U -droplets.) The first of the lemmas is for horizontal crossings.

Lemma 8.15. *Let S be a u -strip, where $u \in \{u^l, u^r\}$.*

(i) *Let $1 \leq \beta_1 \leq \beta_2 \leq \alpha$, and suppose that $\text{IH}(\beta_1, \beta_2)$ holds. If*

$$\pi(S, u) \leq p^{-\beta_1(1-2\eta)-\eta} \quad \text{and} \quad h(S) \leq p^{-\beta_2(1-2\eta)-\eta},$$

then S is u -crossed with probability at most $p^{\delta' \pi(S, u)}$, where $\delta' = \delta'(\beta_1 + \beta_2)$.

(ii) *Suppose that $\text{IH}(\alpha, \alpha + 1)$ holds. If*

$$\pi(S, u) \leq p^{-\alpha(1-2\eta)-\eta} \quad \text{and} \quad h(S) \leq \frac{\xi}{p^\alpha} \log \frac{1}{p},$$

then S is u -crossed with probability at most

$$\exp \left(- p^{O(\xi)} \cdot \pi(S, u) \right).$$

The second of the two lemmas is for vertical crossings. Recall that u^* has difficulty at least $\alpha + 1$, and therefore either $\bar{\alpha}(u^*) \geq \alpha + 1$ or $\alpha^-(u^*) = \infty$. The behaviour of the \mathcal{U} -bootstrap process differs markedly depending on which of these two cases we are in.

If u^* is not a drift direction then the lemma says it is unlikely that a u^* -strip of an appropriate size is u^* -crossed – this is what one would expect. If u^* is a drift direction, then instead the lemma is stated in terms of crossing u -strips, where $\sigma(u) = p^{1-\eta}$. Why might this be the natural direction in which to bound growth? Since u^* is a drift direction, $\alpha^+(u^*)$ may be as small as 1, and therefore one would expect a triangle of sites to form on the u^* -side of the droplet, similarly to the set T in Figure 3. By rotating u^* through an angle of $p^{1-\eta}$, we are ‘giving away’ more sites than one would expect to become infected, but not so many more that it adversely affects the bound. We expand on these remarks before the proof of the lemma.

Note that, while the lemma is stated only for $u \in \mathcal{S}_U^+$, it is plain by symmetry that a similar statement holds for $u \in \mathcal{S}_U^-$.

Lemma 8.16. *Let $u \in \mathcal{S}_U^+$ be such that either*

$$\begin{aligned} u = u^* \quad \text{and} \quad \bar{\alpha}(u^*) \geq \alpha + 1, \quad \text{or} \\ \sigma(u) = p^{1-\eta} \quad \text{and} \quad \alpha^-(u^*) = \infty. \end{aligned}$$

Let $1 \leq \beta_2 \leq \beta_1 \leq \alpha + 1$, with $\beta_2 \neq \alpha + 1$, and suppose that $\text{IH}(\beta_1, \beta_2)$ holds. Let S be a u -strip such that

$$w(S) \leq p^{-\beta_1(1-2\eta)-\eta} \quad \text{and} \quad h(S) \leq p^{-\beta_2(1-2\eta)-\eta}.$$

Then S is u -crossed with probability at most $p^{\delta' \pi(S,u)}$, where $\delta' = \delta'(\beta_1 + \beta_2)$.

The first step towards proving Lemmas 8.15 and 8.16 is a deterministic description of the structure of $S \cap A$ when S is u -crossed, which is given by Lemma 8.19. We partition the u -strip into consecutive u -strips S_1, \dots, S_m of constant u -projection, and we consider how the infection could spread from the $(-u)$ -side of S (and the adjacent half-plane \mathbb{H}_u) to the u side of S . One of the key concepts we use will be that of a ‘ u -weak γ -cluster’, defined as follows.

Definition 8.17. Fix $\beta_1, \beta_2 \in \mathbb{N}$ and let $\kappa_0 = \kappa_0(\beta_1 + \beta_2)$. For each $u \in \mathcal{S}'_U$, define a graph G_{u, κ_0} with vertex set \mathbb{Z}^2 and edge set E , where $\{x, y\} \in E$ if and only if $\|x - y\|_u \leq \kappa_0$. We say that a set of vertices $Z \subset \mathbb{Z}^2$ is u -weakly connected if it is connected in the graph G_{u, κ_0} . Now let $\gamma \in \mathbb{N}$. A u -weak γ -cluster is a set of γ sites that are u -weakly connected.

In what follows we always take $\gamma \leq \bar{\alpha}(u)$, so that u -weak $(\gamma - 1)$ -clusters only cause ‘local’ new infections in \mathbb{H}_u^c .

We can now define the deterministic structural property that we shall prove in Lemma 8.19 is implied by the event that S is u -crossed by A . The definition is illustrated in Figure 5.

Definition 8.18. Fix $\beta_1, \beta_2 \in \mathbb{N}$ and let $\kappa_0 = \kappa_0(\beta_1 + \beta_2)$. Let $u \in \mathcal{S}'_U$ and $\gamma \in \mathbb{N}$, and suppose that S is a u -strip. Set

$$m := \left\lfloor \frac{\pi(S, u)}{3\kappa_0\gamma} \right\rfloor,$$

and let $S_1 \cup \dots \cup S_{m+1}$ be a partition of S into u -strips with $\pi(S_j, u) = 3\kappa_0\gamma$ for each $1 \leq j \leq m$, and $\pi(S_{m+1}, u) < 3\kappa_0\gamma$. A (u, γ) -partition for $S \cap A$ is a sequence (a_1, \dots, a_k) of positive integers with $m = a_1 + \dots + a_k$, such that for each $1 \leq j \leq k$, setting $t_j = a_1 + \dots + a_j$, either

- $a_j = 1$ and $S_{t_j} \cap A$ contains a u -weak γ -cluster, or
- there exists an \mathcal{S}_U -droplet D internally spanned by $(S_{t_{j-1}+1} \cup \dots \cup S_{t_j}) \cap A$, where

$$\max \{w(D), h(D)\} \geq \frac{a_j \kappa_0}{5}.$$

Recall that λ is a sufficiently large fixed constant chosen before each of the $\kappa_0(\beta)$. In particular we have $5\alpha/\eta \leq \lambda \ll \kappa_0(2)$.

Lemma 8.19. Fix $\beta_1, \beta_2 \in \mathbb{N}$ and let $\kappa_0 = \kappa_0(\beta_1 + \beta_2)$. Let $u \in \mathcal{S}'_U$, let S be a u -strip, and let $\gamma \leq \min\{\bar{\alpha}(u), \lambda\}$. Suppose S is u -crossed by A . Then there exists a (u, γ) -partition for $S \cap A$.

Roughly speaking, the proof of the lemma is as follows. We shall show that if $S_1 \cap A$ does not contain a u -weak γ -cluster, then S_1 cannot itself be u -crossed. Since S is u -crossed, this will allow us to deduce that there exists a droplet D internally

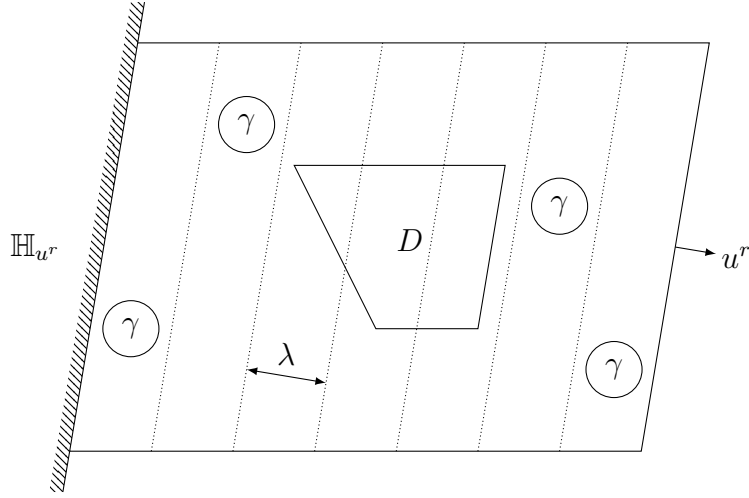


FIGURE 5. A u^r -crossed u^r -strip S together with a possible (u^r, γ) -partition for $S \cap A$ in which $a_1 = a_2 = a_4 = a_5 = 1$ and $a_3 = 3$.

spanned by $S \cap A$ such that $D \cap S_1 \neq \emptyset$, and moreover such that D extends at least halfway across S_1 . We call such a droplet D a *saver*. Letting a_1 be maximal such that $D \cap S_{a_1} \neq \emptyset$, the result follows by induction on m .

Proof of Lemma 8.19. As noted above, the proof is by induction on m . If $m = 0$ there is nothing to prove, so let $m \geq 1$ and assume that the result holds for every smaller non-negative value of m . If S_1 contains a u -weak γ -cluster then we are done, since we may set $a_1 = 1$ and observe that $S \setminus S_1$ is u -crossed by A .

So assume that S_1 does not contain a u -weak γ -cluster, let Y_1, \dots, Y_s be the collection of u -weakly connected components in $S \cap A$ that are each also u -weakly connected to \mathbb{H}_u , and set

$$Y := Y_1 \cup \dots \cup Y_s \quad \text{and} \quad Z := [\mathbb{H}_u \cup Y] \cap S.$$

We claim that $|Y_i| \leq \gamma - 1$ for each $1 \leq i \leq s$. Indeed, if $|Y_i| \geq \gamma$ then there exists a u -weak γ -cluster $Y' \subset Y_i$ all of whose elements are within $\kappa_0 \gamma$ of \mathbb{H}_u in the u -norm. Recalling from (28) that $\langle x, u \rangle \leq \|x\|_u$ for every $x \in \mathbb{Z}^2$, and that S_1 has u -projection $3\kappa_0 \gamma$, it follows that $Y' \subset S_1$. This contradicts our assumption, and thus proves the claim.

In what follows we shall use several times the bound

$$\rho(u, \gamma) \ll \kappa_0, \tag{32}$$

which is a consequence of Lemma 8.12 and the condition $\gamma \leq \lambda$, and the fact that κ_0 was chosen to be sufficiently large.

Next we claim that

$$Z = ([\mathbb{H}_u \cup Y_1] \cup \dots \cup [\mathbb{H}_u \cup Y_s]) \cap S. \tag{33}$$

To prove this, let

$$z_i \in [\mathbb{H}_u \cup Y_i] \setminus \mathbb{H}_u \quad \text{and} \quad z_j \in [\mathbb{H}_u \cup Y_j] \setminus \mathbb{H}_u,$$

and note that $\|z_i - Y_i\|_u \leq \rho(u, \gamma)$ and $\|z_j - Y_j\|_u \leq \rho(u, \gamma)$, by the definition of $\rho(u, \gamma)$. Hence

$$\|z_i - z_j\|_u \geq \kappa_0 - 2\rho(u, \gamma) \geq 2\nu,$$

where the first inequality follows from the triangle inequality, and for the second we use (32). Therefore, the set

$$[\mathbb{H}_u \cup Y_1] \cup \dots \cup [\mathbb{H}_u \cup Y_s]$$

is closed (and contains Y), which proves (33).

We are now ready to prove our key claim, which says that, under our assumption that S_1 does not contain a u -weak γ -cluster, there exists an internally spanned droplet in S that has large intersection with S_1 .

Claim 8.20. *There exists a droplet D internally spanned by $S \cap A$ such that*

$$|\langle D - \mathbb{H}_u, u \rangle| \leq 2\kappa_0\gamma \quad \text{and} \quad \max\{w(D), h(D)\} \geq \frac{\kappa_0}{5}.$$

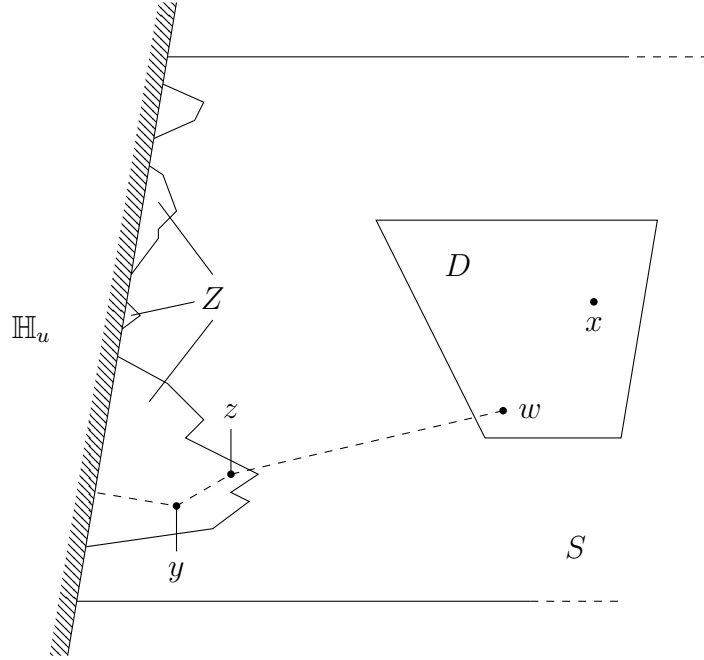


FIGURE 6. The situation in the proof of Claim 8.20 is depicted assuming $z \in Z$. The size of the projection $|\langle D - \mathbb{H}_u, u \rangle|$ is at most the total length of the dashed line in the u -norm (see (36)).

Proof of Claim 8.20. The u -strip S is u -crossed, so there exist sites $z \in \mathbb{H}_u \cup Z$ and $w \in [S \cap A \setminus Y]$ with

$$\|w - z\|_u \leq 2 \cdot \|w - z\| \leq 2\kappa. \quad (34)$$

Indeed, if this were not true then $\mathbb{H}_u \cup Z \cup [S \cap A \setminus Y]$ would be closed and S would not be u -crossed, because $\nu < \kappa$. Let \mathcal{D} be the output of the spanning algorithm with input $S \cap A \setminus Y$, and let $D \in \mathcal{D}$ be the droplet spanned by the strongly connected component of $[S \cap A \setminus Y]$ containing w . If $z \in \mathbb{H}_u$, then it follows by (34) and the u -norm bound in (28) that

$$|\langle D - \mathbb{H}_u, u \rangle| \leq \|w - \mathbb{H}_u\|_u \leq \|w - z\|_u \leq 2\kappa. \quad (35)$$

On the other hand, if $z \in Z$, then $\|z - Y\|_u \leq \rho(u, \gamma) \ll \kappa_0$ by (32), and so since any $y \in Y$ is within distance at most $\gamma\kappa_0$ of \mathbb{H}_u in the u -norm, using the triangle inequality we have

$$|\langle D - \mathbb{H}_u, u \rangle| \leq \|w - \mathbb{H}_u\|_u \leq \|w - z\|_u + \|z - Y\|_u + \gamma\kappa_0 \leq 2\gamma\kappa_0. \quad (36)$$

To bound the dimensions of D , let $x \in D \cap A \setminus Y$, and observe that

$$\|x - (\mathbb{H}_u \cup Y)\|_u \geq \kappa_0$$

by the definition of Y . Hence

$$\|x - (\mathbb{H}_u \cup Z)\|_u \geq \kappa_0 - \rho(u, \gamma),$$

since (33) implies $\|z - Y\|_u \leq \rho(u, \gamma)$ for every $z \in Z$. Therefore we have

$$\|x - (\mathbb{H}_u \cup Z)\| \geq \frac{1}{2} \cdot \|x - (\mathbb{H}_u \cup Z)\|_u \geq \frac{\kappa_0 - \rho(u, \gamma)}{2}.$$

But, by our choice of w (and (34)), we also have

$$\|w - (\mathbb{H}_u \cup Z)\| \leq \|w - z\| \leq \kappa.$$

Hence, since $x, w \in D$, it follows that

$$\begin{aligned} \max\{w(D), h(D)\} &\geq \frac{\|w - x\|}{2} \geq \frac{\|x - (\mathbb{H}_u \cup Z)\| - \|w - (\mathbb{H}_u \cup Z)\|}{2} \\ &\geq \frac{\kappa_0 - \rho(u, \gamma) - 2\kappa}{4} \geq \frac{\kappa_0}{5}, \end{aligned}$$

where again we have used (32). \square

To complete the proof of the lemma, simply set $a_1 = \max\{i : D \cap S_i \neq \emptyset\}$, and observe that $S \setminus (S_1 \cup \dots \cup S_{a_1})$ is u -crossed by A . It follows easily from the claim and our choice of a_1 that $\max\{w(D), h(D)\} \geq a_1\kappa_0/5$, as required. \square

We next prove a simple lemma that gives a bound, depending on the size of S , on the probability that $S \cap A$ admits a (u, γ) -partition. Given $u \in \mathcal{S}'_U$ and a u -strip S , let $g(S)$ be an upper bound on the number of u -weak γ -clusters in a sub-strip $S' \subset S$ of u -projection $3\kappa_0\gamma$.

Lemma 8.21. *Let $1 \leq \beta_1 \leq \alpha + 1$ and $1 \leq \beta_2 \leq \alpha$, and assume that $\text{IH}(\beta_1, \beta_2)$ holds. Let $u \in \mathcal{S}'_U$, and let S be a u -strip such that $|S| = p^{-O(1)}$ and*

$$\pi(S, u) \leq p^{-\beta(1-2\eta)-\eta}, \quad (37)$$

where $\beta := \min\{\beta_1, \beta_2\}$. Then the probability that $S \cap A$ admits a (u, γ) -partition is at most

$$\max_{0 \leq j \leq m} \left(1 - (1 - p^\gamma)^{g(S)}\right)^{m-j} (m \cdot p^{2\alpha})^j, \quad (38)$$

where $m = \lfloor \pi(S, u)/3\kappa_0\gamma \rfloor$.

Proof. We first deal with a small technicality: the saver droplets need only be *internally spanned* by the sites in $S \cap A$; they do not have to be *contained* in S , and therefore their dimensions may be too large to use $\text{IH}(\beta_1, \beta_2)$. Moreover, even if the savers *are* contained in S , they may still have dimensions too large to use $\text{IH}(\beta_1, \beta_2)$. However, neither of these is a problem, as we now show. Let D be any saver droplet (so D is internally spanned by $S \cap A$) such that either

$$w(D) \geq p^{-\beta_1(1-2\eta)-\eta} \quad \text{or} \quad h(D) \geq p^{-\beta_2(1-2\eta)-\eta}. \quad (39)$$

Then by Lemma 6.9, applied once with $u = u^*$ and again if necessary with $u = u^\perp$, there exists a droplet $D' \subset D$, also spanned by sites in S , such that either

$$\begin{aligned} w(D') &\leq p^{-\beta_1(1-2\eta)-\eta} \quad \text{and} \quad p^{-\beta_2(1-2\eta)-\eta}/3 \leq h(D') \leq p^{-\beta_2(1-2\eta)-\eta}, \\ \text{or} \quad p^{-\beta_1(1-2\eta)-\eta}/3 &\leq w(D') \leq p^{-\beta_1(1-2\eta)-\eta} \quad \text{and} \quad h(D') \leq p^{-\beta_2(1-2\eta)-\eta}. \end{aligned}$$

Therefore, by $\text{IH}(\beta_1, \beta_2)$, we have $\mathbb{P}_p(I^\times(D')) \leq p^{\delta k/3}$, where $\delta = \delta(\beta_1 + \beta_2)$ and

$$k := \min \{p^{-\beta_1(1-2\eta)-\eta}, p^{-\beta_2(1-2\eta)-\eta}\}.$$

But $k \geq \pi(S, u)$, since S satisfies (37), and therefore

$$\mathbb{P}_p(I^\times(D')) \leq p^{\delta\pi(S, u)/3}.$$

Hence the probability that S admits a (u, γ) -partition containing a saver droplet D satisfying (39) is at most $p^{-O(1)} \cdot p^{\delta\pi(S, u)/3}$, since there are at most $p^{-O(1)}$ distinct \mathcal{S}_U -droplets spanned by sites in S . We are now done, because if $m = \lfloor \pi(S, u)/3\kappa_0\gamma \rfloor$, then we have

$$p^{\delta\pi(S, u)/3 - O(1)} \leq (m \cdot p^{2\alpha})^m,$$

since κ_0 is sufficiently large, and this is at most (38).

Having dealt with the exceptional saver droplets, we move on to the main part of the proof. Henceforth we assume that if D is a saver droplet in a (u, γ) -partition of S then the dimensions of D do not satisfy (39). (We do not assume, however, that D is necessarily contained in S .)

To start, note that for each $1 \leq i \leq m$, the probability that $S_i \cap A$ contains a u -weak γ -cluster is at most

$$1 - (1 - p^\gamma)^{g(S)}, \quad (40)$$

by Harris's inequality, since by definition there are at most $g(S)$ such sets in S_i .

Next, observe that there are at most $p^{-O(1)}$ distinct \mathcal{S}_U -droplets that are internally spanned by a subset of S , since $|S| = p^{-O(1)}$, and that each such droplet D is internally spanned with probability at most $p^{\delta a}$, by $\text{IH}(\beta_1, \beta_2)$, where $\delta = \delta(\beta_1 + \beta_2)$ and $a = \max\{w(D), h(D)\}$. Thus, for each $1 \leq i \leq j \leq m$, the probability that

$S_i \cup \dots \cup S_j$ contains an internally spanned droplet D with $\max\{w(D), h(D)\} \geq (j - i + 1)\kappa_0/5$ is at most

$$p^{\delta(j-i+1)\kappa_0/5-O(1)} \leq p^{2\alpha(j-i+1)}, \quad (41)$$

since κ_0 was chosen so that $\kappa_0 \gg 1/\delta$.

Finally, note that there are at most m^j partitions of m containing at least $m - j$ ones. By (40) and (41), it follows that S admits a (u, γ) -partition with probability at most

$$\max_{0 \leq j \leq m} \left(1 - (1 - p^\gamma)^{g(S)}\right)^{m-j} (m \cdot p^{2\alpha})^j,$$

as claimed. \square

We shall now apply Lemmas 8.19 and 8.21 three times: once to prove Lemma 8.15 for horizontal crossings, and twice to prove Lemma 8.16 for vertical crossings, once each for drift and non-drift directions. We begin with horizontal crossings.

Proof of Lemma 8.15. Suppose first that $\text{IH}(\beta_1, \beta_2)$ holds, where $1 \leq \beta_1 \leq \beta_2 \leq \alpha$, and let S be a u -strip, where $u \in \{u^l, u^r\}$ and

$$\pi(S, u) \leq p^{-\beta_1(1-2\eta)-\eta} \quad \text{and} \quad h(S) \leq p^{-\beta_2(1-2\eta)-\eta}.$$

If S is u -crossed by A , then, recalling that $\bar{\alpha}(u) \geq \alpha$, it follows by Lemma 8.19 that there exists a (u, α) -partition for $S \cap A$.

Now, noting that there are at most $O(h(S))$ u -weak γ -clusters in each sub-strip $S_i \subset S$, where the implicit constant depends on $\kappa_0 = \kappa_0(\beta_1 + \beta_2)$, it follows from Lemma 8.21 that $S \cap A$ admits a (u, α) -partition with probability at most

$$\max_{0 \leq j \leq m} \left(1 - (1 - p^\alpha)^{O(h(S))}\right)^{m-j} (m \cdot p^{2\alpha})^j, \quad (42)$$

where $m = \lfloor \pi(S, u)/3\kappa_0\gamma \rfloor$. Since $h(S) \leq p^{-\beta_2(1-2\eta)-\eta}$ and $\beta_2 \leq \alpha$, we have

$$1 - (1 - p^\alpha)^{O(h(S))} = O(p^{\alpha-\beta_2(1-2\eta)-\eta}) \leq p^\eta,$$

and since $\pi(S, u) \leq p^{-\beta_1(1-2\eta)-\eta}$ and $\beta_1 \leq \alpha$ we have $m \cdot p^{2\alpha} \leq p^\alpha$. Therefore (42) is at most

$$\max_{0 \leq j \leq m} p^{\eta(m-j)} \cdot p^{\alpha j} = p^{\eta m} \leq p^{\delta' \pi(S, u)},$$

as required, where $\delta' = \delta'(\beta_1 + \beta_2) \leq \eta/6\kappa_0(\beta_1 + \beta_2)\gamma$.

Now suppose that $\text{IH}(\alpha, \alpha + 1)$ holds, and let S be a u -strip, where $u \in \{u^l, u^r\}$ and

$$\pi(S, u) \leq p^{-\alpha(1-2\eta)-\eta} \quad \text{and} \quad h(S) \leq \frac{\xi}{p^\alpha} \log \frac{1}{p}.$$

Then, exactly as above, it follows that (42) is an upper bound on the probability that S is u -crossed by A . Since $h(S) \leq \frac{\xi}{p^\alpha} \log \frac{1}{p}$, we have

$$1 - (1 - p^\alpha)^{O(h(S))} \leq 1 - \exp\left(-O(p^\alpha \cdot h(S))\right) \leq 1 - p^{O(\xi)} \leq e^{-p^{O(\xi)}},$$

so, recalling that $\pi(S, u) = O(m)$, it follows that (42) is at most

$$\max_{0 \leq j \leq m} \left(e^{-p^{O(\xi)}} \right)^{m-j} p^{\alpha j} \leq \exp \left(-p^{O(\xi)} \cdot m \right) = \exp \left(-p^{O(\xi)} \cdot \pi(S, u) \right),$$

as required. \square

Next we use Lemma 8.15 to bound the probability of the event $\Delta(D, D')$ when $w(D') - w(D)$ is not too large. This bound is one of the ingredients we need in order to deduce Lemma 8.2, which is our bound on the probability a critical droplet is internally spanned, from Lemma 8.9, our bound on $\mathbb{P}_p(I^\times(D))$ using hierarchies.

Lemma 8.22. *Let $D \subset D'$ be nested \mathcal{S}_U -droplets.*

(i) *Let $1 \leq \beta_1 \leq \beta_2 \leq \alpha$, and suppose that $\text{IH}(\beta_1, \beta_2)$ holds. If*

$$w(D') - w(D) \leq p^{-\beta_1(1-2\eta)-\eta} \quad \text{and} \quad h(D') \leq p^{-\beta_2(1-2\eta)-\eta},$$

then

$$\mathbb{P}_p(\Delta(D, D')) \leq p^{\Omega(\delta')(w(D')-w(D))},$$

where $\delta' = \delta'(\beta_1 + \beta_2)$ and the constant implicit in $\Omega(\cdot)$ depends only on \mathcal{U} .

(ii) *Suppose that $\text{IH}(\alpha, \alpha + 1)$ holds. If*

$$w(D') - w(D) \leq p^{-\alpha(1-2\eta)-\eta} \quad \text{and} \quad h(D') \leq \frac{\xi}{p^\alpha} \log \frac{1}{p},$$

then

$$\mathbb{P}_p(\Delta(D, D')) \leq \exp \left(-p^{O(\xi)}(w(D') - w(D)) \right).$$

Proof. Let $S^r \subset D'$ be the largest u^r -strip contained in D' such that the $(-u^r)$ -side of S^r intersects the u^r -side of D . Define S^l similarly on the u^l -side of D . Now, if the event $\Delta(D, D')$ occurs, then S^r is u^r -crossed and S^l is u^l -crossed. Since

$$\frac{w(D') - w(D)}{2} \leq \max \{w(S^r), w(S^l)\} \leq w(D') - w(D),$$

and therefore

$$\Omega(w(D') - w(D)) = \max \{ \pi(S^r, u^r), \pi(S^l, u^l) \} \leq w(D') - w(D),$$

it follows by Lemma 8.15 that

$$\mathbb{P}_p(\Delta(D, D')) \leq p^{\Omega(\delta')(w(D')-w(D))}$$

under the assumptions of part (i), and that

$$\mathbb{P}_p(\Delta(D, D')) \leq \exp \left(-p^{O(\xi)}(w(D') - w(D)) \right)$$

under the assumptions of part (ii), as required. \square

The proof of Lemma 8.16, which bounds the probability of vertical crossings, is conceptually more difficult. When $\bar{\alpha}(u^*) \geq \alpha + 1$ (the ‘non-drift’ case), the proof is straightforward. Indeed, in this case the application of Lemma 8.19 is the same as in the proof of Lemma 8.15. When $\alpha^-(u^*) = \infty$ (the ‘drift’ case), and $u \in \mathcal{S}_U^+$ is such that $\sigma(u) = p^{1-\eta}$, this naive approach no longer works. Instead we use the stretched geometry of the u -norm to control the unbounded sideways growth of small sets. This is the only point in the proof of Theorem 8.1 where we specifically need the u -norm.

Proof of Lemma 8.16. We may assume throughout that $u \in \mathcal{S}_U^+$ and that S is a u -crossed u -strip such that

$$w(S) \leq p^{-\beta_1(1-2\eta)-\eta} \quad \text{and} \quad h(S) \leq p^{-\beta_2(1-2\eta)-\eta}. \quad (43)$$

We begin with the easier ‘non-drift’ case; the proof is almost identical to that of Lemma 8.15.

Case 1: $\bar{\alpha}(u^*) > \alpha$. In this case we have $u = u^*$, so S is a u^* -crossed u^* -strip. By Lemma 8.19, there exists a $(u^*, \alpha + 1)$ -partition for $S \cap A$.

Now, noting that there are at most $O(w(S))$ u^* -weak γ -clusters in each constant-height sub-strip $S_i \subset S$, it follows from Lemma 8.21 that $S \cap A$ admits a $(u^*, \alpha + 1)$ -partition with probability at most

$$\max_{0 \leq j \leq m} \left(1 - (1 - p^{\alpha+1})^{O(w(S))}\right)^{m-j} (m \cdot p^{2\alpha})^j, \quad (44)$$

where $m = \lfloor h(S)/3\kappa_0\gamma \rfloor$. Since $w(S) \leq p^{-\beta_1(1-2\eta)-\eta}$ and $\beta_1 \leq \alpha + 1$, we have

$$1 - (1 - p^{\alpha+1})^{O(w(S))} = O(p^{\alpha+1-\beta_1(1-2\eta)-\eta}) \leq p^\eta,$$

and since $h(S) \leq p^{-\beta_2(1-2\eta)-\eta}$ and $\beta_2 \leq \alpha$ we have $m \cdot p^{2\alpha} \leq p^\alpha$. Therefore (44) is at most

$$\max_{0 \leq j \leq m} p^{\eta(m-j)} \cdot p^{\alpha j} = p^{\eta m} \leq p^{\delta' h(S)} = p^{\delta' \pi(S, u)},$$

as required, where $\delta' = \delta'(\beta_1 + \beta_2) \leq \eta/6\kappa_0(\beta_1 + \beta_2)\gamma$.

We now turn to the ‘drift’ case.

Case 2: $\alpha^-(u^*) = \infty$. In this case $u \in \mathcal{S}_U^+$ is such that $\sigma(u) = p^{1-\eta}$, and we again assume that S is a u -crossed u -strip satisfying (43). We apply Lemma 8.19 with $\gamma = 5\alpha/\eta$, which is permissible because $\bar{\alpha}(u) = \infty$ and $5\alpha/\eta < \lambda$. The lemma implies the existence of a (u, γ) -partition for $S \cap A$. We shall use Lemma 8.21 to bound the probability that such a partition occurs.

In order to apply the lemma, we need an upper bound $g(S)$ on the number of u -weak γ -clusters in a sub-strip of S of u -projection $3\kappa_0\gamma$. Since u (and hence the norm $\|\cdot\|_u$) depends on p , the number of such clusters is not $O(w(S))$: there are still $O(w(S))$ choices for the first site in the u -weak γ -cluster, but there are now as many as $O(1/\sigma)$ choices for each of the remaining $\gamma - 1$ sites. Therefore, setting $g(S) = w(S) \cdot p^{-\gamma(1-\eta)}$, we obtain a suitable upper bound.

By Lemma 8.21, it follows that there exists a (u, γ) -partition for $S \cap A$ with probability at most

$$\max_{0 \leq j \leq m} \left(1 - (1 - p^\gamma)^{g(S)}\right)^{m-j} (m \cdot p^{2\alpha})^j, \quad (45)$$

where $m := \lfloor \pi(S, u)/3\kappa_0\gamma \rfloor$. Now, since

$$1 - (1 - p^\gamma)^{g(S)} \leq w(S) \cdot p^{-\gamma(1-\eta)} \cdot p^\gamma \leq w(S) \cdot p^{5\alpha} \leq p^{3\alpha},$$

by our choice of γ , it follows that (45) is at most

$$\max_{0 \leq j \leq m} p^{3\alpha(m-j)} \cdot p^{\alpha j} \leq p^{\alpha m} \leq p^{\delta' \pi(S, u)},$$

where $\delta' = \delta'(\beta_1 + \beta_2) \leq \alpha/3\kappa_0\gamma$. This completes the proof of the lemma. \square

8.4. The induction steps. In this subsection we prove the induction steps. The following lemma will be used to deduce the implications $\text{IH}(\beta, \beta) \Rightarrow \text{IH}(\beta + 1, \beta)$ and $\text{IH}(\beta + 1, \beta) \wedge \text{IH}(\beta, \beta + 1) \Rightarrow \text{IH}(\beta + 1, \beta + 1)$.

For convenience, we shall occasionally use the notation $\exp_p(x) := p^x$.

Lemma 8.23. *Let $1 \leq \beta_1 \leq \alpha$ and $1 \leq \beta_2 \leq \alpha$ with $\beta_1 \geq \beta_2 - 1$, and suppose that $\text{IH}(\beta_1, \beta_2)$ holds. Let D be an \mathcal{S}_U -droplet such that*

$$p^{-\beta_1(1-2\eta)-\eta} \leq w(D) \leq p^{-(\beta_1+1)(1-2\eta)-\eta},$$

and $h(D) \leq p^{-\beta_2(1-2\eta)-\eta}$. Then

$$\mathbb{P}_p(I^\times(D)) \leq p^{\Omega(\delta')w(D)}.$$

where $\delta' = \delta'(\beta_1 + \beta_2)$ and the constant implicit in $\Omega(\cdot)$ depends only on \mathcal{U} .

Proof. We shall use the hierarchies framework from Section 8.1 with $\beta = \beta_1$. To begin, recall the bound from Lemma 8.9:

$$\mathbb{P}_p(I^\times(D)) \leq \sum_{\mathcal{H} \in \mathcal{H}_D} \left(\prod_{u \in L(\mathcal{H})} \mathbb{P}_p(I^\times(D_u)) \right) \left(\prod_{u \rightarrow v} \mathbb{P}_p(\Delta(D_v, D_u)) \right). \quad (46)$$

In order to use this bound, we need estimates for the probability that a seed is internally spanned, the probability of the event $\Delta(D_v, D_u)$, and the number of good hierarchies for D .

First, for each $u \in L(\mathcal{H})$ we have $w(D_u) \leq p^{-\beta_1(1-2\eta)-\eta}$ by Definition 8.6 (v). Also, $D_u \subset D$, so $h(D) \leq p^{-\beta_2(1-2\eta)-\eta}$. Hence, by $\text{IH}(\beta_1, \beta_2)$,

$$\mathbb{P}_p(I^\times(D_u)) \leq p^{\delta w(D_u)}, \quad (47)$$

where $\delta = \delta(\beta_1 + \beta_2)$. Second, we bound the probability of the event $\Delta(D_v, D_u)$ using Lemma 8.22. The droplets D_u and D_v satisfy $w(D_u) - w(D_v) \leq p^{-\beta_1(1-2\eta)-\eta}$, by Definition 8.6, and $h(D_v) \leq h(D_u) \leq p^{-\beta_2(1-2\eta)-\eta}$, since D_u is contained in D . Hence, Lemma 8.22 states that

$$\mathbb{P}_p(\Delta(D_v, D_u)) \leq p^{\Omega(\delta')(w(D_u) - w(D_v))}, \quad (48)$$

where $\delta' = \delta'(\beta_1 + \beta_2)$. Thus, combining these first two estimates with (46), and using the notation $\exp_p(x) := p^x$, we obtain

$$\mathbb{P}_p(I^\times(D)) \leq \sum_{\mathcal{H} \in \mathcal{H}_D} \exp_p \left(\delta \sum_{u \in L(\mathcal{H})} w(D_u) + \Omega(\delta') \sum_{u \rightarrow v} (w(D_u) - w(D_v)) \right). \quad (49)$$

We now divide into two cases according to the number of big seeds of \mathcal{H} . The idea is as follows: if the number of big seeds is small then there are few enough hierarchies (by Lemma 8.11) that we can uniformly bound the probability of each; if the number of big seeds is large then the contribution to (49) from the large seeds alone outweighs the combinatorial cost of counting the good hierarchies.

Note first that if $\mathcal{H} \in \mathcal{H}_D$ has at least b big seeds, then

$$\sum_{u \in L(\mathcal{H})} w(D_u) \geq \frac{b \cdot p^{-\beta_1(1-2\eta)-\eta}}{3}. \quad (50)$$

Now set $B := p^{-1+2\eta}$. We claim that if $\mathcal{H} \in \mathcal{H}_D$ has at most B big seeds, then

$$\sum_{u \in L(\mathcal{H})} w(D_u) + \sum_{u \rightarrow v} (w(D_u) - w(D_v)) \geq w(D) - O(|V(G_{\mathcal{H}})|) \geq \frac{w(D)}{2}. \quad (51)$$

The first inequality in (51) is an easy consequence of Definition 8.5 and the geometric inequality

$$w(D_1 \cup D_2) \leq w(D_1) + w(D_2) + O(1),$$

which holds for any connected pair of droplets D_1 and D_2 . The second inequality follows since

$$|V(G_{\mathcal{H}})| = O(B \cdot w(D) \cdot p^{\beta_1(1-2\eta)+\eta}) = o(w(D))$$

for every $\beta_1 \geq 1$, which follows from (25), (26) and the definition of B .

Combining (50) and (51) with (49), we obtain

$$\mathbb{P}_p(I^\times(D)) \leq \sum_{b \leq B} |\mathcal{H}_D^b| \cdot p^{\Omega(\delta')w(D)} + \sum_{b > B} |\mathcal{H}_D^b| \cdot \exp_p \left(\Omega(\delta') \cdot b \cdot p^{-\beta_1(1-2\eta)-\eta} \right), \quad (52)$$

where here we have used $\delta \gg \delta'$. In order to bound $|\mathcal{H}_D^b|$, suppose first that $b \leq B$. Then, by Lemma 8.11, we have

$$|\mathcal{H}_D^b| \leq \exp_p \left(-O(b \cdot w(D) \cdot p^{\beta_1(1-2\eta)+\eta}) \right) \leq e^{w(D)}. \quad (53)$$

On the other hand, if $b > B$, then

$$|\mathcal{H}_D^b| \leq \exp_p \left(-O(b \cdot w(D) \cdot p^{\beta_1(1-2\eta)+\eta}) \right) \leq \exp_p \left(-O(b \cdot p^{-1+2\eta}) \right), \quad (54)$$

since $w(D) \leq p^{-(\beta_1+1)(1-2\eta)-\eta}$. Finally, inserting (53) and (54) into (52), we obtain

$$\mathbb{P}_p(I^\times(D)) \leq B \cdot p^{\Omega(\delta')w(D)} + \sum_{b > B} \exp_p \left(\Omega(\delta') \cdot b \cdot p^{-\beta_1(1-2\eta)-\eta} \right) \leq p^{\Omega(\delta')w(D)},$$

as required, where we used that $B = e^{o(w(D))}$. \square

We need one more lemma for vertical crossings. The lemma says that if a droplet D is ‘crossed’ with help from *both* the u^* -side and the $(-u^*)$ -side, then, in a certain sense (which is made explicit in the lemma), the droplet is at least half ‘crossed’ with help from just one side. In Lemma 8.25, this will allow us to transfer from a droplet that is ‘crossed’ with help from both sides to a u -crossed u -strip, for an appropriate u . One such application of the lemma (in which some of the labelling is different) is shown in Figure 7.

Lemma 8.24. *Let D_{u^*} , D and D_{-u^*} be disjoint \mathcal{S}_U -droplets such that their union $D_{u^*} \cup D \cup D_{-u^*}$ is also an \mathcal{S}_U -droplet, and D_{u^*} , D and D_{-u^*} partition the union horizontally, so the u^* -side of D_{u^*} touches the $(-u^*)$ -side of D , and similarly for D and D_{-u^*} . Suppose that*

$$Z := [D_{u^*} \cup (D \cap A) \cup D_{-u^*}] \quad (55)$$

contains a strongly connected component Z' such that $D_{u^} \cup D_{-u^*} \subset Z'$. Then there exists a set $L \subset D$ with $h(L) \geq h(D)/2 - \kappa$ such that, for some $u \in \{u^*, -u^*\}$, $L \cup D_u$ is a strongly connected component of $[D_u \cup (D \cap A)]$.*

Proof. For each $u \in \{u^*, -u^*\}$, let Z_u be the strongly connected component of $[D_u \cup (D \cap A)]$ containing D_u . If the set $Z_{u^*} \cup Z_{-u^*}$ is strongly connected, then for each $u \in \{u^*, -u^*\}$ let $L_u := Z_u \cap D$, and observe that

$$h(L_{u^*}) + h(L_{-u^*}) \geq h(D) - \kappa,$$

as required.

So suppose that $Z_{u^*} \cup Z_{-u^*}$ is not strongly connected, and let \mathcal{Y} be the collection of strongly connected components of $[(D \cap A) \setminus (Z_{u^*} \cup Z_{-u^*})]$. Then $Z_u \cup Y$ is not strongly connected for any $u \in \{u^*, -u^*\}$ and $Y \in \mathcal{Y}$, and thus

$$\mathcal{Y} \cup \{Z_{u^*}, Z_{-u^*}\}$$

is precisely the collection of strongly connected components of Z . But Z contains a strongly connected component containing both D_{u^*} and D_{-u^*} , and so this is a contradiction, which completes the proof of the lemma. \square

The next lemma, which deals with vertical crossings, will be used to deduce the implications $\text{IH}(\beta, \beta) \Rightarrow \text{IH}(\beta, \beta + 1)$ and $\text{IH}(\beta + 1, \beta) \wedge \text{IH}(\beta, \beta + 1) \Rightarrow \text{IH}(\beta + 1, \beta + 1)$, and in the proof of Lemma 8.2.

Lemma 8.25. *Assume $\text{IH}(\beta_1, \beta_2)$, where $\beta_1 \leq \alpha + 1$, $\beta_2 \leq \alpha$ and $\beta_2 \geq \beta_1 - 1$. Let D be an \mathcal{S}_U -droplet such that*

$$w(D) \leq p^{-\beta_1(1-2\eta)-\eta} \quad \text{and} \quad h(D) \geq p^{1-2\eta} \cdot w(D).$$

Then

$$\mathbb{P}_p(I^\times(D)) \leq p^{\delta' h(D)/7},$$

where $\delta' = \delta'(\beta_1 + \beta_2)$.

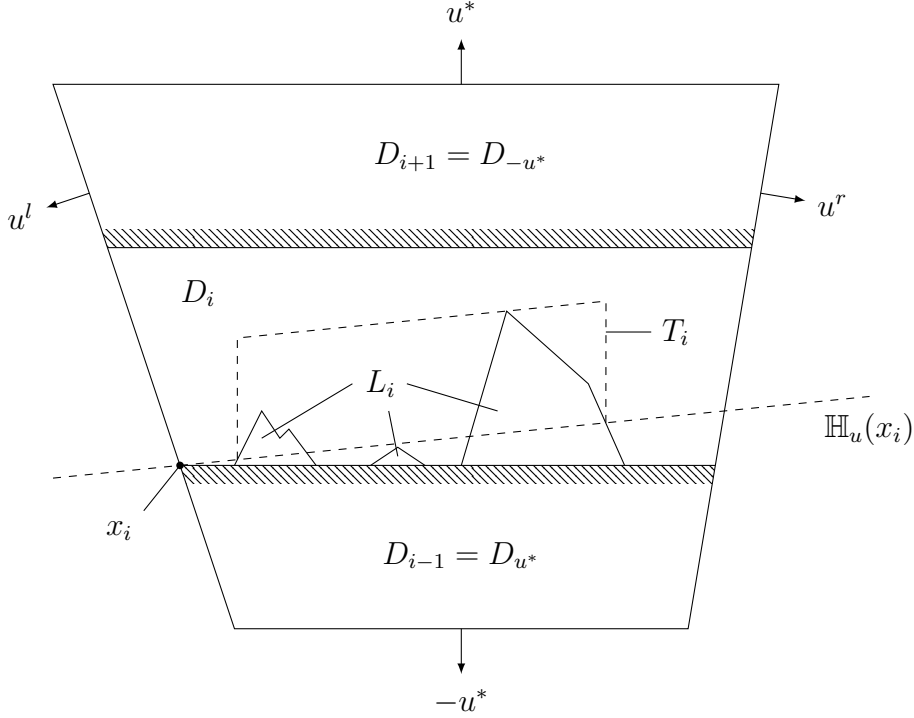


FIGURE 7. The figure depicts the application of Lemma 8.24 in the proof of Lemma 8.25 assuming $\alpha^-(u^*) = \infty$. Included are the \mathcal{S}_U -droplets D_{i-1} , D_i and D_{i+1} ; the set L_i , which is such that $D_{i-1} \cup L_i$ is a strongly connected component of $[D_{i-1} \cup (D_i \cap A)]$; the site x_i at the bottom-left corner of D_i ; the half plane $\mathbb{H}_u(x_i)$, where $\sigma(u) = p^{1-\eta}$; and the minimal u -strip T_i such that $L_i \subset T_i \cup \mathbb{H}_u(x_i)$, which is u -crossed.

Proof. We partition D horizontally into droplets D_1, \dots, D_m (thus, consecutive pairs have touching $\pm u^*$ -sides), where

$$12 \cdot p^{1-\eta} \cdot w(D) \leq h(D_i) \leq 13 \cdot p^{1-\eta} \cdot w(D)$$

for each $1 \leq i \leq m$. This is possible because $h(D) \geq p^{1-2\eta} \cdot w(D)$. Since D is internally spanned, for each $1 \leq i \leq m$ it follows that condition (55) of Lemma 8.24 is satisfied if we set $D_{u^*} = D_{i-1}$, $D_{-u^*} = D_{i+1}$, and if the droplet ‘ D ’ in (55) is taken to be our D_i (we interpret D_0 as being an additional droplet below D_1). Therefore there exists a set $L_i \subset D_i$ with $h(L_i) \geq h(D_i)/2 - \kappa$ such that either $L_i \cup D_{i-1}$ is a strongly connected component of $[D_{i-1} \cup (D_i \cap A)]$ or $L_i \cup D_{i+1}$ is a strongly connected component of $[D_{i+1} \cup (D_i \cap A)]$. Let E_i denote the event that such a set L_i exists.

Suppose E_i occurs and let us assume that $L_i \cup D_{i-1}$ is a strongly connected component of $[D_{i-1} \cup (D_i \cap A)]$. We again divide into two cases according to whether or not u^* is a drift direction. Suppose first that $\bar{\alpha}(u^*) > \alpha$, and observe

that the minimal u^* -strip S_i containing L_i is u^* -crossed. Now, we have $w(S_i) = w(L_i) \leq p^{-\beta_1(1-2\eta)-\eta}$, and

$$h(S_i) \leq 4 \cdot p^{1-\eta} \cdot w(D) = o(p^{-\beta_2(1-2\eta)-\eta}),$$

since $h(D_i)/2 - \kappa \leq h(S_i) \leq h(D_i)$ and $\beta_2 \geq \beta_1 - 1$. Therefore, by Lemma 8.16, each such u^* -strip S_i is u^* -crossed with probability at most $p^{\delta' h(S_i)}$.

Now suppose $\alpha^-(u^*) = \infty$. Let x_i be the element of \mathbb{R}^2 at the intersection of the u^l - and $(-u^*)$ -sides of D_i . Let T_i be the minimal u -strip such that

$$L_i \subset \mathbb{H}_u(x_i) \cup T_i,$$

where $u \in \mathcal{S}_U^+$ is such that $\sigma(u) = p^{1-\eta}$. Observe that T_i is u -crossed by $D_i \cap A$: this is because

$$L_i \subset [\mathbb{H}_u(x_i) \cup (T_i \cap A)],$$

since $L_i \cup D_{i-1}$ is a strongly connected component of $[D_{i-1} \cup (D_i \cap A)]$, and because $L_i \cap \partial(T_i, u) \neq \emptyset$ (see Figure 7). As in the previous case, the dimensions of T_i satisfy the conditions of Lemma 8.16, and therefore T_i is u -crossed with probability at most $p^{\delta' \pi(T_i, u)} \leq p^{\delta' h(T_i)/2}$, where we are using the inequalities

$$3 \cdot h(T_i) \geq h(D_i) \geq 12 \cdot p^{1-\eta} \cdot w(D) \geq 12 \cdot p^{1-\eta} \cdot w(T_i),$$

to give

$$\pi(T_i, u) \geq h(T_i) - 2 \cdot \sigma \cdot w(T_i) \geq \frac{h(T_i)}{2}.$$

In either case, there are at most $p^{-O(1)}$ choices for the strips S_i or T_i , and there were two choices for L_i at the start, so we conclude that $\mathbb{P}_p(E_i) \leq p^{\delta' h(D_i)/6}$. The events E_1, \dots, E_m are independent, since E_i depends only on the set $D_i \cap A$, and so

$$\mathbb{P}_p(I^\times(D)) \leq \exp_p \left(\frac{\delta'}{6} \sum_{i=1}^m h(D_i) \right) \leq p^{\delta' h(D)/7},$$

and this completes the proof of the lemma. \square

We are now ready to prove Lemma 8.4.

Proof of Lemma 8.4. We shall prove by induction on $\beta_1 + \beta_2$ that $\text{IH}(\beta_1, \beta_2)$ holds for every pair $(\beta_1, \beta_2) \in \mathbb{N}^2$ with

$$2 \leq \beta_1 + \beta_2 \leq 2\alpha + 1 \quad \text{and} \quad |\beta_1 - \beta_2| \leq 1.$$

To begin, note that $\text{IH}(1, 1)$ follows from Lemma 6.11, since $\delta(1, 1)$ was chosen sufficiently small (depending on η). The induction step is split into three claims.

Claim 8.26. *For each $1 \leq \beta \leq \alpha$ we have*

$$\text{IH}(\beta, \beta) \Rightarrow \text{IH}(\beta + 1, \beta).$$

Proof. Let D be a droplet with

$$w(D) \leq p^{-(\beta+1)(1-2\eta)-\eta} \quad \text{and} \quad h(D) \leq p^{-\beta(1-2\eta)-\eta}.$$

We are required to show that $\mathbb{P}_p(I^\times(D)) \leq p^{\delta \max\{w(D), h(D)\}}$, where $\delta = \delta(2\beta + 1)$. Note that if $w(D) \leq p^{-\beta(1-2\eta)-\eta}$ then this follows immediately from $\text{IH}(\beta, \beta)$ (since $\delta(2\beta + 1) \leq \delta(2\beta)$), so we may assume that $w(D) \geq p^{-\beta(1-2\eta)-\eta}$. Now, applying Lemma 8.23 with $\beta_1 = \beta_2 = \beta$, it follows that

$$\mathbb{P}_p(I^\times(D)) \leq p^{\Omega(\delta'(2\beta))w(D)} \leq p^{\delta w(D)},$$

as required, since we may ensure that $\delta = \delta(2\beta + 1) \ll \delta'(2\beta)$. \square

Claim 8.27. *For each $1 \leq \beta \leq \alpha$ we have*

$$\text{IH}(\beta, \beta) \Rightarrow \text{IH}(\beta, \beta + 1).$$

Proof. Let D be a droplet with

$$w(D) \leq p^{-\beta(1-2\eta)-\eta} \quad \text{and} \quad h(D) \leq p^{-(\beta+1)(1-2\eta)-\eta}.$$

As in the previous claim, we must show that $\mathbb{P}_p(I^\times(D)) \leq p^{\delta \max\{w(D), h(D)\}}$, where now $\delta = \delta(2\beta + 1)$. Once again, if $h(D) \leq p^{-\beta(1-2\eta)-\eta}$ then this follows immediately from $\text{IH}(\beta, \beta)$, so we may assume that $h(D) \geq p^{-\beta(1-2\eta)-\eta}$. Applying Lemma 8.25 with $\beta_1 = \beta_2 = \beta$, we have

$$\mathbb{P}_p(I^\times(D)) \leq p^{\delta'(2\beta)h(D)/7} \leq p^{\delta h(D)},$$

as required, since we may choose $\delta = \delta(2\beta + 1) \leq \delta'(2\beta)/7$. \square

Claim 8.28. *For each $1 \leq \beta \leq \alpha - 1$ we have*

$$(\text{IH}(\beta + 1, \beta) \wedge \text{IH}(\beta, \beta + 1)) \Rightarrow \text{IH}(\beta + 1, \beta + 1).$$

Proof. Let D be an \mathcal{S}_U -droplet with

$$w(D) \leq p^{-(\beta+1)(1-2\eta)-\eta} \quad \text{and} \quad h(D) \leq p^{-(\beta+1)(1-2\eta)-\eta}.$$

For this final claim we must show that $\mathbb{P}_p(I^\times(D)) \leq p^{\delta \max\{w(D), h(D)\}}$, where $\delta = \delta(2\beta + 2)$. If $\min\{w(D), h(D)\} \leq p^{-\beta(1-2\eta)-\eta}$, then this follows from either $\text{IH}(\beta, \beta + 1)$ or $\text{IH}(\beta + 1, \beta)$, so we may assume that $\min\{w(D), h(D)\} \geq p^{-\beta(1-2\eta)-\eta}$.

Suppose first that $w(D) \geq h(D)$. Then, applying Lemma 8.23 with $\beta_1 = \beta$ and $\beta_2 = \beta + 1$, it follows that

$$\mathbb{P}_p(I^\times(D)) \leq p^{\Omega(\delta'(2\beta+1))w(D)} \leq p^{\delta w(D)},$$

as required, since we may choose $\delta = \delta(2\beta + 2) \ll \delta'(2\beta + 1)$.

On the other hand, if $w(D) \leq h(D)$ then, applying Lemma 8.25 with $\beta_1 = \beta + 1$ and $\beta_2 = \beta$, it follows that

$$\mathbb{P}_p(I^\times(D)) \leq p^{\delta'(2\beta)h(D)/7} \leq p^{\delta h(D)},$$

again as required, where we ensure that $\delta = \delta(2\beta + 2) \leq \delta'(2\beta + 1)/7$. \square

Together with $\text{IH}(1, 1)$, these claims imply $\text{IH}(\alpha + 1, \alpha)$ and $\text{IH}(\alpha, \alpha + 1)$, which completes the proof of the lemma. \square

8.5. Internally spanned critical droplets. In this subsection we complete the proof of Lemma 8.2, using the method of hierarchies, Lemma 8.4, and the lemmas from the previous subsection. Recall from Definition 2.5 that we call an \mathcal{S}_U -droplet D *critical* if one of the following holds:

- (T) $w(D) \leq p^{-\alpha-1/5}$ and $\frac{\xi}{p^\alpha} \log \frac{1}{p} \leq h(D) \leq \frac{3\xi}{p^\alpha} \log \frac{1}{p}$, or
- (L) $p^{-\alpha-1/5} \leq w(D) \leq 3p^{-\alpha-1/5}$ and $h(D) \leq \frac{\xi}{p^\alpha} \log \frac{1}{p}$,

where ξ is a sufficiently small positive constant. We shall show that the lemma holds with

$$\delta = \frac{\xi \cdot \delta'(2\alpha + 1)}{7}.$$

Proof of Lemma 8.2. Let D be a critical droplet, and suppose first that D is of type (T). Then, recalling that $\eta = (10\alpha)^{-1}$, we have

$$w(D) \leq p^{-(\alpha+1)(1-2\eta)-\eta} \quad \text{and} \quad h(D) \geq p^{-1+2\eta} w(D).$$

Now observe that we may apply Lemma 8.25 with $\beta_1 = \alpha + 1$ and $\beta_2 = \alpha$, since $\text{IH}(\alpha + 1, \alpha)$ holds by Lemma 8.4. This gives

$$\mathbb{P}_p(I^\times(D)) \leq p^{\delta'(2\alpha+1)h(D)/7} \leq \exp\left(-\frac{\delta}{p^\alpha} \left(\log \frac{1}{p}\right)^2\right),$$

as required, since $\delta \leq \xi \cdot \delta'(2\alpha + 1)/7$.

So suppose from now on that D is of type (L). The proof of the lemma in this case is similar to the proof of Lemma 8.23, but, in contrast to type (T) droplets above (and the corresponding Lemma 8.25), we cannot use Lemma 8.23 directly, because the bound on $\mathbb{P}_p(\Delta(D_v, D_u))$ given by Lemma 8.22 takes a different form under these conditions.

We again apply the hierarchies framework, as in the proof of Lemma 8.23, but with $\beta = \alpha$. Recall that

$$\mathbb{P}_p(I^\times(D)) \leq \sum_{\mathcal{H} \in \mathcal{H}_D} \left(\prod_{u \in L(\mathcal{H})} \mathbb{P}_p(I^\times(D_u)) \right) \left(\prod_{u \rightarrow v} \mathbb{P}_p(\Delta(D_v, D_u)) \right), \quad (56)$$

by Lemma 8.9. First, if $u \in L(\mathcal{H})$ then $w(D_u) \leq p^{-\alpha(1-2\eta)-\eta}$ and $h(D_u) \leq h(D)$, so $\text{IH}(\alpha, \alpha + 1)$ implies that

$$\mathbb{P}_p(I^\times(D_u)) \leq p^{\delta(2\alpha+1)w(D_u)}. \quad (57)$$

Moreover if $u \rightarrow v$ then $w(D_u) - w(D_v) \leq p^{-\alpha(1-2\eta)-\eta}$ and $h(D_v) \leq h(D_u) \leq h(D)$, so by Lemma 8.22 (ii) and $\text{IH}(\alpha, \alpha + 1)$ (the latter holding by Lemma 8.4), we have

$$\mathbb{P}_p(\Delta(D_v, D_u)) \leq \exp\left(-p^{O(\xi)}(w(D_u) - w(D_v))\right). \quad (58)$$

Combining these bounds with (56), we obtain

$$\mathbb{P}_p(I^\times(D)) \leq \sum_{\mathcal{H} \in \mathcal{H}_D} \exp \left[-p^{O(\xi)} \left(\sum_{u \in L(\mathcal{H})} w(D_u) + \sum_{u \rightarrow v} (w(D_u) - w(D_v)) \right) \right], \quad (59)$$

since $\delta(2\alpha + 1) \log(1/p) > p^{O(\xi)}$.

Setting $B := p^{-2/3}$, we divide into two cases according to whether there are ‘few’ (at most B) or ‘many’ big seeds. If $\mathcal{H} \in \mathcal{H}_D^b$ has at least b big seeds, then

$$\sum_{u \in L(\mathcal{H})} w(D_u) \geq \frac{b \cdot p^{-\alpha(1-2\eta)-\eta}}{3}, \quad (60)$$

and if $\mathcal{H} \in \mathcal{H}_D$ has at most B big seeds, then

$$\sum_{u \in L(\mathcal{H})} w(D_u) + \sum_{u \rightarrow v} (w(D_u) - w(D_v)) \geq w(D) - O(|V(G_{\mathcal{H}})|) \geq \frac{w(D)}{2}, \quad (61)$$

as in (51). The first inequality follows exactly as before, and the second inequality holds since

$$|V(G_{\mathcal{H}})| = O(B \cdot w(D) \cdot p^{\alpha(1-2\eta)+\eta}) = o(w(D))$$

by (25) and (26). By Lemma 8.11, we have

$$|\mathcal{H}_D^b| \leq \exp_p \left(-O(b \cdot w(D) \cdot p^{\alpha(1-2\eta)+\eta}) \right) \leq \begin{cases} e^{w(D)} & \text{if } b \leq B \\ \exp_p \left(-O(b \cdot p^{-1+2\eta}) \right) & \text{otherwise,} \end{cases}$$

since $w(D) \leq p^{-(\alpha+1)(1-2\eta)-\eta}$. Inserting this bound and the bounds from (60) and (61) into the bound on $\mathbb{P}_p(I^\times(D))$ in (59), we obtain

$$\begin{aligned} \mathbb{P}_p(I^\times(D)) &\leq \sum_{b \leq B} \exp \left(-p^{O(\xi)} \cdot w(D) \right) + \sum_{b > B} \exp \left(-O(b \cdot p^{-\alpha(1-2\eta)-\eta/2}) \right) \\ &\leq \exp \left(-p^{O(\xi)} \cdot w(D) \right) \leq \exp \left(-p^{-\alpha-1/6} \right), \end{aligned}$$

as required, since $w(D) \geq p^{-\alpha-1/5}$ and ξ is sufficiently small. \square

8.6. The occupation time of the origin. Having established Lemma 8.2, we now have all the tools we need to complete the proof of Theorem 8.1. From here, the deduction is straightforward, and similar to the corresponding part of the proof for balanced families in Section 7. In short, if the origin is infected by time t , then either there is an internally spanned critical droplet within distance t of the origin, or the origin itself is contained in a small internally spanned droplet. We bound the probability of the former event using Lemma 8.2, and that of the latter using Lemma 8.4. As in Section 7, we write $D(k)$ for the unique minimal \mathcal{S}_U -droplet all of whose sides are at ℓ_2 distance at least λk from the origin, where $k \in \mathbb{N}$ and λ is a large fixed constant.¹⁰

¹⁰Of course, in this section droplets are \mathcal{S}_U -droplets, while in Section 7 they were \mathcal{S}_B -droplets, so the definition of $D(k)$ is not exactly the same. We trust this will cause the reader no confusion.

Lemma 8.29. *Let $\varepsilon > 0$ be sufficiently small, and let $t \in \mathbb{N}$ and $p \in (0, 1)$ be such that*

$$p^\alpha \left(\log \frac{1}{p} \right)^{-2} \log t \leq \varepsilon.$$

Then, with high probability as $p \rightarrow 0$, there does not exist an internally spanned critical droplet $D \subset D(t)$.

Proof. By Lemma 8.2, the probability that a given critical droplet D is internally spanned is at most

$$\exp \left(-\frac{\delta}{p^\alpha} \left(\log \frac{1}{p} \right)^2 \right),$$

for some constant $\delta > 0$. Since there are at most $O(t^2 p^{-O(1)}) = t^{O(1)}$ critical droplets in $D(t)$, it follows that the probability in question is at most

$$t^{O(1)} \exp \left(-\frac{\delta}{p^\alpha} \left(\log \frac{1}{p} \right)^2 \right) \rightarrow 0$$

as $t \rightarrow \infty$, as required, since ε is sufficiently small. \square

Lemma 8.30. *Let $\varepsilon > 0$ be sufficiently small, and let $t \in \mathbb{N}$ and $p \in (0, 1)$ be such that*

$$p^\alpha \left(\log \frac{1}{p} \right)^{-2} \log t \leq \varepsilon.$$

Then, with high probability as $p \rightarrow 0$, there does not exist an internally spanned \mathcal{S}_U -droplet $D \subset D(t)$, with

$$w(D) \leq p^{-\alpha-1/5} \quad \text{and} \quad h(D) \leq \frac{\xi}{p^\alpha} \log \frac{1}{p}, \quad (62)$$

such that D contains the origin.

Proof. For each $w, h \in \mathbb{N}$, let $E(w, h)$ denote the event that the origin is contained in an internally spanned droplet of width w and height h . Suppose first that $\max\{w, h\} \leq p^{-\alpha(1-2\eta)-\eta}$, and observe that in this case, by Lemma 8.4, we have

$$\mathbb{P}_p(E(w, h)) \leq wh \cdot p^{\delta \max\{w, h\}},$$

where $\delta = \delta(2\alpha)$, since there are at most wh droplets of width w and height h that contain the origin. On the other hand, if $\max\{w, h\} \geq p^{-\alpha(1-2\eta)-\eta}$, then, by Lemma 6.9 (applied first with $u = u^*$, then with $u = u^\perp$), there exists an internally spanned droplet $D' \subset D$ with $k \leq \max\{w(D'), h(D')\} \leq 3k$, where $k = p^{-\alpha(1-2\eta)-\eta}/3$. By Lemma 8.4, it follows that

$$\mathbb{P}_p(E(w, h)) \leq p^{-O(1)} \cdot p^{\delta k},$$

since there are at most $p^{-O(1)}$ droplets of width at most w and height at most h that contain the origin.

Hence, the probability that there exists an internally spanned droplet $D \subset D(t)$, containing the origin and satisfying (62), is at most

$$\sum_{w=1}^{3k} \sum_{h=1}^{3k} wh \cdot p^{\delta \max\{w,h\}} + p^{-O(1)} \cdot p^{\delta k} \rightarrow 0$$

as $t \rightarrow \infty$, as claimed. \square

We can now finally deduce our main result, Theorem 8.1.

Proof of Theorem 8.1. Let $\varepsilon > 0$ be a sufficiently small constant, and set

$$p^\alpha \left(\log \frac{1}{p} \right)^{-2} \log t = \varepsilon.$$

We shall show that $\tau > t$ with high probability as $p \rightarrow 0$.

Observe first that if $\tau \leq t$ then $\mathbf{0} \in [D(t) \cap A]$, since otherwise there would have to be a path of successive infections from $D(t)^c$ to the origin, and any such path has length greater than t . Let $\mathcal{D} = \langle D(t) \cap A \rangle$ be the output of the spanning algorithm with input $D(t) \cap A$, and let $D \in \mathcal{D}$ be the internally spanned droplet such that $\mathbf{0} \in [D \cap A]$.

Now, by Lemma 8.30, the probability that such a droplet exists satisfying (62) is $o(1)$. On the other hand, if there exists an internally spanned droplet D with

$$w(D) \geq p^{-\alpha-1/5} \quad \text{or} \quad h(D) \geq \frac{\xi}{p^\alpha} \log \frac{1}{p},$$

then by Lemma 6.9 (applied once with $u = u^*$, then again if necessary with $u = u^\perp$) there exists an internally spanned critical droplet $D' \subset D$. But, by Lemma 8.29, the probability that such a droplet exists is also $o(1)$. It follows that $\tau > t$ with high probability, and this completes the proof of the theorem. \square

9. CONJECTURES FOR HIGHER DIMENSIONS

We conclude by briefly discussing the \mathcal{U} -bootstrap percolation models in higher dimensions. Fix an integer $d \geq 2$ and let \mathcal{U} be a d -dimensional update family. The definition of the stable set $\mathcal{S} = \mathcal{S}(\mathcal{U})$ is the natural generalization of the two-dimensional definition:

$$\mathcal{S} := \{u \in S^{d-1} : [\mathbb{H}_u^d] = \mathbb{H}_u^d\},$$

where

$$\mathbb{H}_u^d := \{x \in \mathbb{Z}^d : \langle x, u \rangle < 0\}$$

is the discrete half-space in \mathbb{Z}^d with normal $u \in S^{d-1}$. Observe that, as in two dimensions, it is easy to show that the dichotomy $[\mathbb{H}_u^d] \in \{\mathbb{H}_u^d, \mathbb{Z}^d\}$ holds for any unit vector $u \in S^{d-1}$.

Let $\mu : \mathcal{L}(S^{d-1}) \rightarrow \mathbb{R}$ denote the Lebesgue measure on the collection of Lebesgue-measurable subsets of S^{d-1} . Generalizing Definition 1.1, we classify d -dimensional update families as follows:

Definition 9.1. A d -dimensional update family is:

- (i) *subcritical* if $\mu(C \cap \mathcal{S}) > 0$ for every hemisphere $C \subset S^{d-1}$;
- (ii) *critical* if there exists a hemisphere $C \subset S^{d-1}$ such that $\mu(C \cap \mathcal{S}) = 0$ and if $C \cap \mathcal{S} \neq \emptyset$ for every open hemisphere $C \subset S^{d-1}$;
- (iii) *supercritical* if $C \cap \mathcal{S} = \emptyset$ for some open hemisphere $C \subset \mathcal{S}$.

As in two dimensions, the subcritical/critical/supercritical trichotomy depends only on the stable set \mathcal{S} . However, unlike in two dimensions, we expect there to be a further subdivision of critical families into $d - 1$ classes according to the value of r for which the model behaves (broadly) like the classical r -neighbour model.

Conjecture 9.2. Let \mathcal{U} be a d -dimensional bootstrap percolation update family.

- (i) If \mathcal{U} is subcritical then $\liminf_{n \rightarrow \infty} p_c(\mathbb{Z}_n^d, \mathcal{U}) > 0$.
- (ii) If \mathcal{U} is critical then there exist $r \in \{1, \dots, d - 1\}$ and $\alpha \in \mathbb{Q}$ such that

$$p_c(\mathbb{Z}_n^d, \mathcal{U}) = \left(\frac{1}{\log_{(r)} n} \right)^{\alpha + o(1)}.$$

- (iii) If \mathcal{U} is supercritical then $p_c(\mathbb{Z}_n^d, \mathcal{U}) = n^{-\Theta(1)}$.

Of the three statements in the conjecture, that for supercritical families is likely to be relatively straightforward, while that for subcritical families was originally made by Balister, Bollobás, Przykucki and Smith [2]. For critical families, one might hope to prove an even sharper result, along the lines of Theorems 1.4 and 1.5, but even the much weaker bounds conjectured above appear to be far out of reach with current techniques.

ACKNOWLEDGEMENT

This work was started during a visit of the second author to IMPA in the spring of 2013. The author would like to thank IMPA for its hospitality.

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